Nonuniform (h, k, μ, ν)-dichotomy and Stability of Nonautonomous Discrete Dynamics

Jimin Zhang[a], Meng Fan[b], Liu Yang[a]

[a]*School of Mathematical Sciences, Heilongjiang University, 74 Xuefu Street, Harbin, Heilongjiang, 150080, P. R. China*

[b]*School of Mathematics and Statistics, Northeast Normal University, 5268 Renmin Street, Changchun, Jilin, 130024, P. R. China*

Abstract

In this paper, a new notion called the general nonuniform (h, k, μ, ν)-dichotomy for a sequence of linear operators is proposed, which occurs in a more natural way and is related to nonuniform hyperbolicity. Then, sufficient criteria are established for the existence of nonuniform (h, k, μ, ν)-dichotomy in terms of appropriate Lyapunov exponents for the sequence of linear operators. Moreover, we investigate the stability theory of sequences of nonuniformly hyperbolic linear operators in Banach spaces, which admit a nonuniform (h, k, μ, ν)-dichotomy. In the case of linear perturbations, we investigate parameter dependence of robustness or roughness of the nonuniform (h, k, μ, ν)-dichotomies and show that the stable and unstable subspaces of nonuniform (h, k, μ, ν)-dichotomies for the linear perturbed system are Lipschitz continuous for the parameters. In the case of nonlinear perturbations, we construct a new version of the Grobman-Hartman theorem and explore the existence of parameter dependence of stable Lipschitz invariant manifolds when the nonlinear perturbation is of Lipschitz type.

Keywords: Nonuniform (h, k, μ, ν)-dichotomies; Roughness; Hartman-Grobman theorem; Stable Lipschitz invariant manifolds
2010 Mathematics Subject Classifications: 34D09, 34D10, 37D25

1. Introduction

The classical notion of the uniform exponential dichotomy, essentially introduced in the seminal work of Perron [1], has been playing a center role in a substantial part of the theory of uniformly hyperbolic dynamical systems. The theory of exponential dichotomies and its applications are widely developed. We refer to the books [2, 3, 4, 5] for more details and references. However, the uniform exponential dichotomy is very stringent for the dynamics and it is of interest and is very important to look for more general types of hyperbolic behavior [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

☆M. Fan is supported by NSFC-10971022, NCET-08-0755, and FRFCU-10JCXK003; J. Zhang and L. Yang are supported by NSFC-11201128, NSFHLJ-A201414, China Postdoctoral Science Foundation, Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province(No.2014TD005), the Heilongjiang University Fund for Distinguished Yonng Scholars (JCL201203).

Email address: mfan@nenu.edu.cn (Jimin Zhang[a], Meng Fan[b], Liu Yang[a])

The concept of nonuniform hyperbolicity, describing the theory of continuous or discrete dynamical systems with nonzero Lyapunov exponents, generalizes the classical concept of uniform hyperbolicity and has been widely recognized both in various fields of mathematics and in practical applications [18, 19, 20, 21]. Recently, various different kinds of nonuniform dichotomy are proposed, which are exhibited by a large class of differential or difference equations and closely related to the theory of nonuniform hyperbolicity, e.g. nonuniform exponential dichotomy [6, 8, 9, 10, 11, 16, 17], nonuniform polynomial dichotomy [13, 22, 23], ρ-nonuniform exponential dichotomy [24], nonuniform (μ, ν)-dichotomy [14, 15, 25, 26, 27], and so on. Moreover, the uniform or nonuniform dichotomy, together with its variants and extensions, is always one of the most important and useful means in the study of the stability theory of the uniform or nonuniform hyperbolic dynamical systems, such as, the roughness in the finite dimensional spaces [2, 7, 26, 28, 29, 30, 31] or in the infinite dimensional spaces [6, 23, 24, 25, 32, 33, 34, 35, 36, 37], the linearization theory [6, 35, 38, 39, 40, 41, 42, 43, 44], and the existence of invariant manifolds and their absolute continuity[6, 45, 46, 47, 48, 49, 50].

In previous studies of uniform or nonuniform dichotomies, the growth rates are always assumed to be the same type of functions. However, the nonuniformly hyperbolic dynamical systems vary greatly in forms and none of the nonuniform dichotomy can well characterize all the nonuniformly hyperbolic dynamics. For example, if we choose some appropriate Lyapunov exponents, then the growth rates may be completely different (see Section 2 below). It is necessary and reasonable to look for more general types of nonuniform dichotomies to explore the dynamics of the nonuniformly hyperbolic dynamical systems.

The nonuniform dichotomy is not only an essential part of the theory of nonuniform hyperbolicity but also an important approach to explore the nonuniform hyperbolicity of dynamical systems. The main novelty of the present work is that we consider a new notion called the generalized nonuniform (h, k, μ, ν)-dichotomy for sequences of nonuniformly hyperbolic linear operators, which not only incorporates the existing notions of the uniform or nonuniform dichotomies as special cases, but also allows the different growth rates in the stable space and unstable space or in the uniform part and nonuniform part with rates of expansion and contraction varying in different manner. Particularly, we will establish a sufficient criterion for sequences of linear operators in block form in a finite-dimensional space to have a nonuniform (h, k, μ, ν)-dichotomy in terms of appropriate Lyapunov exponents in Section 2. It follows from the results in the present paper that the notion of nonuniform (h, k, μ, ν)-dichotomy occurs naturally.

For a nonautonomous discrete dynamics defined by a sequence of linear operators in a Banach space, we investigate the parameter dependence of the roughness of nonuniform (h, k, μ, ν)-dichotomy under sufficiently small linear perturbations in Section 3. With the help of nonuniform (h, k, μ, ν)-dichotomy, we explore the topological conjugacies by establishing a new version of the Grobman-Hartman theorem in Section 4, and, finally, we establish the existence of parameter dependence of stable Lipschitz invariant manifolds.

2. Nonuniform (h, k, μ, ν)-dichotomies

Let $\mathcal{B}(X)$ be the space of bounded linear operators in a Banach space X. Consider the

sequence of invertible linear operators $\{A_m\}_{m\in\mathbb{Z}} \subset \mathcal{B}(X)$. Define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n, & \text{if } m > n, \\ \mathrm{id}, & \text{if } m = n, \\ A_m^{-1}\cdots A_{n-1}^{-1}, & \text{if } m < n. \end{cases}$$

Definition 2.1. *A sequence of numbers $\{u_m\}_{m\in\mathbb{Z}}$ is said to be a growth rate if $\cdots < u_n < \cdots < u_{-1} < u_0 = 1 < u_1 < \cdots < u_m < \cdots$, $\lim_{m\to+\infty} u_m = +\infty, \quad \lim_{n\to-\infty} u_n = 0$.*

Denote by Δ the set of growth rates and always assume that

$$\{h_m\}_{m\in\mathbb{Z}},\ \{k_m\}_{m\in\mathbb{Z}},\ \{\mu_m\}_{m\in\mathbb{Z}},\ \{\nu_m\}_{m\in\mathbb{Z}} \in \Delta$$

throughout the paper.

Definition 2.2. *The sequence of linear operators $(A_m)_{m\in\mathbb{Z}}$ is said to have a* nonuniform (h,k,μ,ν)-dichotomy *if there exist projections P_n for $n \in \mathbb{Z}$ such that*

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n, \quad m,n \in \mathbb{Z}$$

and there exist constants $a < 0 \le b$, $\varepsilon \ge 0$ and $K > 0$ such that

$$\begin{aligned} \|\mathcal{A}(m,n)P_n\| &\le K(h_m/h_n)^a \mu_{|n|}^{\varepsilon}, \qquad m \ge n, \\ \|\mathcal{A}(m,n)Q_n\| &\le K(k_n/k_m)^{-b} \nu_{|n|}^{\varepsilon}, \qquad m \le n, \end{aligned} \tag{2.1}$$

where $Q_n = \mathrm{id} - P_n$ are the complementary projections.

Remark 2.1. *The nonuniform (h,k,μ,ν)-dichotomy is general enough to include as special cases the uniform exponential dichotomy ($h_m = k_m = e^m, \varepsilon = 0$) [51, 39, 52, 42], (h,h)-dichotomy ($h_m = k_m, \varepsilon = 0$) [30], (h,k)-dichotomy ($\varepsilon = 0$) [30], nonuniform exponential dichotomy ($h_m = k_m = e^m, \mu_{|m|} = \nu_{|m|} = e^{|m|}$) [6], nonuniform polynomial dichotomy ($h_m = k_m = \mu_m = \nu_m = m+1, m \in \mathbb{Z}^+$) [23], nonuniform (μ,ν)-dichotomy ($h_m = k_m = \mu_m$ and $\mu_m = \nu_m = \nu_m$, $m \in \mathbb{N}$) [14, 27], ρ-nonuniform exponential dichotomy ($h_m = k_m = \mu_m = \nu_m = e^{\rho(m)}, m \in \mathbb{N}$) [24, 38].*

Remark 2.2. *In [53], the authors proposed a general dichotomy on $\mathbb{N}$ and choose two functions in the stable space and unstable space. While, in Definition 2.2, four different functions for growth rates are chosen in the stable space, the unstable space, the uniform part, and the nonuniform part. Compared with the notion in [53], Definition 2.2 is more reasonable and occurs in a more natural way, where a and b play the role of Lyapunov exponents and ε measures the nonuniformity of dichotomies. The reason is that, in a finite-dimensional space, one can establish a sufficient criterion for sequences of linear operators in block form to have a nonuniform (h,k,μ,ν)-dichotomy in terms of appropriate Lyapunov exponents, and Definition 2.2 can more closely connect the theory of Lyapunov exponents with the theory of nonuniform hyperbolicity. Those facts will be found in the following discussion.*

Example 2.1. *Consider the difference equation in* $\mathbb{R}^2$

$$z^1_{m+1} = \left(\frac{h_{m+1}}{h_m}\right)^{-\theta_1} e^{\theta_2 d^1_m} z^1_m, \qquad z^2_{m+1} = \left(\frac{k_{m+1}}{k_m}\right)^{\theta_3} e^{\theta_2 d^2_m} z^2_m, \quad m \in \mathbb{Z}, \tag{2.2}$$

where

$$\begin{aligned} d^1_m &= \log(\mu_{m+1})(\sin\log(\mu_{m+1}) - 1) + \cos\log(\mu_{m+1}) - \cos\log(\mu_m) \\ &\quad - \log(\mu_m)(\sin\log(\mu_m) - 1), \\ d^2_m &= \log(\nu_{m+1})(\sin\log(\nu_{m+1}) - 1) + \cos\log(\nu_{m+1}) - \cos\log(\nu_m) \\ &\quad - \log(\nu_m)(\sin\log(\nu_m) - 1) \end{aligned}$$

and $\theta_1, \theta_2, \theta_3$ *are positive constants.*

Set $P_m(z^1_m, z^2_m) = z^1_m$ and $Q_m(z^1_m, z^2_m) = z^2_m$ for $m \in \mathbb{Z}$. Then

$$\begin{aligned} \mathcal{A}(m,n)P_n &= \left(\frac{h_m}{h_n}\right)^{-\theta_1} e^{\theta_2 d^1(m,n)}, \\ \mathcal{A}(m,n)Q_n &= \left(\frac{k_m}{k_n}\right)^{\theta_3} e^{\theta_2 d^2(m,n)}, \end{aligned}$$

where

$$\begin{aligned} d^1(m,n) &= \log(\mu_m)(\sin\log(\mu_m) - 1) + \cos\log(\mu_m) \\ &\quad - \cos\log(\mu_n) - \log(\mu_n)(\sin\log(\mu_n) - 1), \\ d^2(m,n) &= \log(\nu_m)(\sin\log(\nu_m) - 1) + \cos\log(\nu_m) \\ &\quad - \cos\log(\nu_n) - \log(\nu_n)(\sin\log(\nu_n) - 1). \end{aligned}$$

It follows that

$$\begin{aligned} \|\mathcal{A}(m,n)P_n\| &\le e^{2\theta_2}\left(\frac{h_m}{h_n}\right)^{-\theta_1} \mu_n^{2\theta_2} \le e^{2\theta_2}\left(\frac{h_m}{h_n}\right)^{-\theta_1} \mu_{|n|}^{2\theta_2}, \ m \ge n, \\ \|\mathcal{A}(m,n)Q_n\| &\le e^{2\theta_2}\left(\frac{k_n}{k_m}\right)^{-\theta_3} \nu_n^{2\theta_2} \le e^{2\theta_2}\left(\frac{k_n}{k_m}\right)^{-\theta_3} \nu_{|n|}^{2\theta_2}, \ m \le n \end{aligned}$$

which implies that (2.2) admits a nonuniform (h, k, μ, ν)-dichotomy with

$$K = e^{2\theta_2}, \quad a = -\theta_1, \quad b = \theta_3, \quad \varepsilon = 2\theta_2.$$

In addition to the existing uniform or nonuniform dichotomies, when h, k, μ, ν are chosen to be different sequences, one obtains some new nonuniform dichotomies such as

- $h_m = k_m = \omega_1 m$, $\mu_m = \nu_m = \omega_2 m + 1$ with ω_1, ω_2 being positive constants and $m \in \mathbb{N}$;
- $h_m = k_m = e^m$, $\mu_m = \nu_m = e^{m^3+m}$;
- $h_m = k_m = m + 1$ and $\mu_m = \nu_m = e^m$;

- $h_m = \mu_m = m+1$ and $k_m = \nu_m = e^m$.

Example 2.1 shows the generality of the nonuniform (h, k, μ, ν)-dichotomy. In the following, we establish some sufficient criteria for the sequences of linear operators in block form in a finite-dimensional space to have a nonuniform (h, k, μ, ν)-dichotomy on $\mathbb{N}$.

Assume that $X = \mathbb{R}^n = E \oplus F$ with $\dim E = l$ and $\dim F = n - l$. Given a sequence of invertible matrixes $\{A_m\}_{m\in\mathbb{N}} \subset \mathcal{B}(X)$ with $A_m = \mathrm{diag}(C_m, D_m)$ with respect to the above decomposition. Define $\varphi, \bar{\varphi} : E \to [-\infty, +\infty]$ and $\psi, \bar{\psi} : F \to [-\infty, +\infty]$ by

$$\begin{aligned}
\varphi(y) &= \limsup_{m\to+\infty} \frac{\log \|C_{m-1}\cdots C_1 y\|}{\log h_m},\\
\psi(z) &= \limsup_{m\to+\infty} \frac{\log \|D_{m-1}\cdots D_1 z\|}{\log k_m},\\
\bar{\varphi}(y) &= \limsup_{m\to+\infty} \frac{\log \|(C_1^T\cdots C_{m-1}^T)^{-1} y\|}{\log \bar{h}_m},\\
\bar{\psi}(z) &= \limsup_{m\to+\infty} \frac{\log \|(D_1^T\cdots D_{m-1}^T)^{-1} z\|}{\log \bar{k}_m},
\end{aligned} \tag{2.3}$$

where $y \in E$, $z \in F$, $\bar{h}_m$ and $\bar{k}_m$ are growth rates, and $\log 0 = -\infty$. By carrying out similar arguments to those of Proposition 10.2 in [6] or of Proposition 1 in [22], we claim that

(i) $\varphi(0) = \bar{\varphi}(0) = \psi(0) = \bar{\psi}(0) = -\infty$;

(ii) $\varphi(\tilde{c}y) = \varphi(y)$, $\bar{\varphi}(\tilde{c}y) = \bar{\varphi}(y)$, $\psi(\tilde{c}z) = \psi(z)$ and $\bar{\psi}(\tilde{c}z) = \bar{\psi}(z)$ for $y \in E, z \in F$ and $\tilde{c} \in \mathbb{R} \setminus \{0\}$;

(iii) $\varphi(y'+y'') \le \max\{\varphi(y'), \varphi(y'')\}$, $\bar{\varphi}(y'+y'') \le \max\{\bar{\varphi}(y'), \bar{\varphi}(y'')\}$, $\psi(z'+z'') \le \max\{\psi(z'), \psi(z'')\}$ and $\bar{\psi}(z'+z'') \le \max\{\bar{\psi}(z'), \bar{\psi}(z'')\}$ for $y', y'' \in E$ and $z', z'' \in F$.

(iv) $\varphi(y'+y'') = \max\{\varphi(y'), \varphi(y'')\}$, $\bar{\varphi}(y'+y'') = \max\{\bar{\varphi}(y'), \bar{\varphi}(y'')\}$, $\psi(z'+z'') = \max\{\psi(z'), \psi(z'')\}$ and $\bar{\psi}(z'+z'') = \max\{\bar{\psi}(z'), \bar{\psi}(z'')\}$ whenever $\varphi(y') \ne \varphi(y'')$, $\bar{\varphi}(y') \ne \bar{\varphi}(y'')$, $\psi(z') \ne \psi(z'')$ and $\bar{\psi}(z') \ne \bar{\psi}(z'')$;

(v) $y^1, \cdots, y^m$ are linearly independent if $\varphi(y^1), \cdots, \varphi(y^m)$ or $\bar{\varphi}(y^1), \cdots, \bar{\varphi}(y^m)$ are distinct for $y^1, \cdots, y^m \in E\setminus\{0\}$; $z^1, \cdots, z^{m'}$ are linearly independent if $\psi(z^1), \cdots, \psi(z^{m'})$ or $\bar{\psi}(z^1), \cdots, \bar{\psi}(z^{m'})$ are distinct for $z^1, \cdots, z^{m'} \in F \setminus \{0\}$;

(vi) φ ($\bar{\varphi}$) has at most $r \le l$ ($\bar{r} \le l$) distinct values in $E \setminus \{0\}$, say $-\infty \le \lambda_1 < \cdots < \lambda_r \le +\infty$ ($-\infty \le \bar{\lambda}_{\bar{r}} < \cdots < \bar{\lambda}_1 \le +\infty$); ψ ($\bar{\psi}$) has at most $r' \le n-l$ ($\bar{r}' \le n-l$) distinct values in $F \setminus \{0\}$, say $-\infty \le \chi_1 < \cdots < \chi_{r'} \le +\infty$ ($-\infty \le \bar{\chi}_{\bar{r}'} < \cdots < \bar{\chi}_1 \le +\infty$);

(vii) $E_i = \{y \in E : \lambda(y) \le \lambda_i\}$ ($\overline{E}_i = \{y \in E : \bar{\lambda}(y) \le \bar{\lambda}_i\}$) is a linear space for $i = 1, \cdots, r$ ($i = 1, \cdots, \bar{r}$); $F_i = \{z \in F : \chi(z) \le \chi_i\}$ ($\overline{F}_i = \{z \in F : \bar{\chi}(z) \le \bar{\chi}_i\}$) is a linear space for $i = 1, \cdots, r'$ ($i = 1, \cdots, \bar{r}'$).

If (i), (ii) and (iii) hold, (φ, ψ) ($(\bar{\varphi}, \bar{\psi})$) is said to be the (h, k) ($(\bar{h}, \bar{k})$) Lyapunov exponent with respect to the linear operators $(A_m)_{m\in\mathbb{N}}$. Let $\varrho_1, \cdots, \varrho_n$ and $\zeta_1, \cdots, \zeta_n$ be two bases of $\mathbb{R}^n$, they are said to be dual if $(\varrho_i, \zeta_j) = \omega_{ij}$ for every i, j, where $(\cdot, \cdot)$ is the standard inner

product in $\mathbb{R}^n$ and ω_{ij} is the Kronecker symbol. In order to introduce the regularity coefficients of $\varphi, \bar{\varphi}$ and $\psi, \bar{\psi}$, assume that $\lambda_i, \bar{\lambda}_i, \chi_i, \bar{\chi}_i$ are finite. Define the *regularity coefficient* of φ and $\bar{\varphi}$ by

$$\gamma(\varphi, \bar{\varphi}) = \min \max\{\varphi(\delta_i) + \bar{\varphi}(\bar{\delta}_i) : 1 \leq i \leq l\},$$

where the minimum is taken over all dual bases $\delta_1, \cdots, \delta_l$ and $\bar{\delta}_1, \cdots, \bar{\delta}_l$ of E. The *regularity coefficient of ψ and $\bar{\psi}$* is defined by

$$\bar{\gamma}(\psi, \bar{\psi}) = \min \max\{\psi(\epsilon_i) + \bar{\psi}(\bar{\epsilon}_i) : 1 \leq i \leq n - l\},$$

where the minimum is taken over all dual bases $\epsilon_1, \cdots, \epsilon_{n-l}$ and $\bar{\epsilon}_1, \cdots, \bar{\epsilon}_{n-l}$ of F.

Theorem 2.1. *Assume that $\varphi(y) < 0$ for $y \in E \setminus \{0\}$ and $\psi(z) > 0$ for $z \in F \setminus \{0\}$ with $\lambda_r < 0 < \chi_1$. Then, for any sufficiently small $\tilde{\varepsilon} > 0$, the sequence of linear operators $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform (h, k, μ, ν)-dichotomy with*

$$a = \lambda_r + \tilde{\varepsilon},\ b = \chi_1 + \tilde{\varepsilon},\ \varepsilon = \max\{\gamma(\varphi, \bar{\varphi}),\ \bar{\gamma}(\psi, \bar{\psi})\} + \tilde{\varepsilon},\ \mu_m = h_m \bar{h}_m,\ \nu_m = k_m \bar{k}_m.$$

Proof. Let $\hat{m}_j = \varphi(\hat{y}_1^j)$ and $\check{n}_j = \bar{\varphi}(\check{y}_1^j)$ for $j = 1, \cdots, l$, where $\hat{y}_{m-1}^1, \cdots, \hat{y}_{m-1}^l$ are the columns of $C_{m-1} \cdots C_1$ and $\check{y}_{m-1}^1, \cdots, \check{y}_{m-1}^l$ are the columns of $(C_1^T \cdots C_{m-1}^T)^{-1}$. By (2.3), for any $\tilde{\varepsilon} > 0$, it is not difficult to show that there exists a sufficiently large $\bar{K}_1$ such that

$$\|\hat{y}_{m-1}^j\| \leq \bar{K}_1 h_m^{\hat{m}_j + \tilde{\varepsilon}} \quad \text{and} \quad \|\check{y}_{m-1}^j\| \leq \bar{K}_1 \bar{h}_m^{\check{n}_j + \tilde{\varepsilon}} \tag{2.4}$$

for $m \in \mathbb{N}$ and $j = 1, \cdots, l$. Moreover, from $(\hat{y}_{m-1}^i, \check{y}_{m-1}^j) = \omega_{ij}$ and

$$\left[(C_1^T \cdots C_{m-1}^T)^{-1}\right]^T (C_{m-1} \cdots C_1) = \text{id},$$

it follows that $\gamma(\varphi, \bar{\varphi}) = \max\{\hat{m}_j + \check{n}_j : j = 1, \cdots, l\}$. For $m \geq n$, let

$$\begin{aligned}\mathcal{C}(m, n) &= C_{m-1} \cdots C_n = C_{m-1} \cdots C_n C_{n-1} \cdots C_1 (C_{n-1} \cdots C_1)^{-1} \\ &= (C_{m-1} \cdots C_1) \left[(C_1^T \cdots C_{n-1}^T)^{-1}\right]^T\end{aligned}$$

and $c_{i\tau}(m, n) = \sum_{j=1}^{l} \hat{y}_{m-1}^{ij} \check{y}_{n-1}^{\tau j}$ be the entries of $\mathcal{C}(m, n)$. By (2.4), one has

$$\begin{aligned}|c_{i\tau}(m, n)| &\leq \sum_{j=1}^{l} |\hat{y}_{m-1}^{ij}| |\check{y}_{n-1}^{\tau j}| \leq \sum_{j=1}^{l} \|\hat{y}_{m-1}^j\| \|\check{y}_{n-1}^j\| \\ &\leq \sum_{j=1}^{l} \bar{K}_1^2 h_m^{\hat{m}_j + \tilde{\varepsilon}} \bar{h}_n^{\check{n}_j + \tilde{\varepsilon}} \\ &\leq \sum_{j=1}^{l} \bar{K}_1^2 (h_m / h_n)^{\hat{m}_j + \tilde{\varepsilon}} h_n^{\hat{m}_j + \tilde{\varepsilon}} \bar{h}_n^{\check{n}_j + \tilde{\varepsilon}} \\ &\leq \bar{K}_1^2 l (h_m / h_n)^{\lambda_r + \tilde{\varepsilon}} (h_n \bar{h}_n)^{\gamma(\varphi, \bar{\varphi}) + \tilde{\varepsilon}}\end{aligned}$$

and

$$\|\mathcal{C}(m,n)\xi\|^2 = \left\|\sum_{i=1}^{l}\sum_{\tau=1}^{l} l_\tau c_{i\tau}(m,n)e_i\right\|^2$$
$$\leq \sum_{i=1}^{l}\left(\sum_{\tau=1}^{l} l_\tau^2 \sum_{\tau=1}^{l} c_{i\tau}(m,n)^2\right) \leq \sum_{i=1}^{l}\sum_{\tau=1}^{l} c_{i\tau}(m,n)^2,$$

where $\xi = \sum_{\tau=1}^{l} l_\tau e_\tau$ with $\|\xi\|^2 = \sum_{\tau=1}^{l} l_\tau^2 = 1$ and $\{e_1, \cdots, e_l\}$ is the standard orthogonal basis of E. Therefore,

$$\|\mathcal{C}(m,n)\| \leq \left(\sum_{i=1}^{l}\sum_{\tau=1}^{l} c_{i\tau}(m,n)^2\right)^{1/2}$$
$$\leq \bar{K}_1^2 l^2 (h_m/h_n)^{\lambda_r+\tilde{\varepsilon}}(h_n\bar{h}_n)^{\gamma(\varphi,\bar{\varphi})+\tilde{\varepsilon}} \leq \bar{K}_1^2 l^2 (h_m/h_n)^a \mu_n^\varepsilon.$$

Proceeding similarly to the above arguments, we conclude that there exists a constant $\bar{K}_2$ such that

$$\|\mathcal{D}(m,n)\| = \|D_m^{-1}\cdots D_{n-1}^{-1}\|$$
$$\leq \bar{K}_2^2 (n-l)^2 (k_n/k_m)^{-(\chi_1+\tilde{\varepsilon})}(k_n\bar{k}_n)^{\bar{\gamma}(\psi,\bar{\psi})+\tilde{\varepsilon}}$$
$$\leq \bar{K}_2^2 (n-l)^2 (k_n/k_m)^{-b}\nu_n^\varepsilon.$$

The proof is complete. ∎

In the above discussion, a relatively strong assumption is that A_m is of block form. In fact, we can also establish the existence of nonuniform $(h, k, \mu.\nu)$-dichotomies for more general sequences of linear operators. For example, let

$$\widetilde{\mathcal{A}}(m,n) = \begin{cases} \widetilde{\mathcal{A}}_{m-1}\cdots\widetilde{\mathcal{A}}_n, & \text{if } m > n, \\ \text{id}, & \text{if } m = n, \\ \widetilde{\mathcal{A}}_m^{-1}\cdots\widetilde{\mathcal{A}}_{n-1}^{-1}, & \text{if } m < n \end{cases}$$

and

$$\widetilde{\mathcal{B}}(m,n) = \begin{cases} \widetilde{\mathcal{B}}_{m-1}\cdots\widetilde{\mathcal{B}}_n, & \text{if } m > n, \\ \text{id}, & \text{if } m = n, \\ \widetilde{\mathcal{B}}_m^{-1}\cdots\widetilde{\mathcal{B}}_{n-1}^{-1}, & \text{if } m < n, \end{cases}$$

where $\{\widetilde{\mathcal{A}}_m\}_{m\in\mathbb{Z}}, \{\widetilde{\mathcal{B}}_m\}_{m\in\mathbb{Z}} \subset \mathcal{B}(X)$ are invertible matrix sequences and $\widetilde{\mathcal{B}}_m$ have a block form. $\widetilde{\mathcal{A}}(m,n)$ is said to be *reducible* if there exist a sequence of invertible matrixes $\{\widetilde{S}_m\}_{m\in\mathbb{Z}}$ and a constant $\widetilde{M} > 0$ such that

$$\widetilde{\mathcal{A}}_m\widetilde{S}_m = \widetilde{S}_{m+1}\widetilde{\mathcal{B}}_m, \quad \|S_m\| \leq \widetilde{M}, \quad \|S_m^{-1}\| \leq \widetilde{M}.$$

It is not difficult to show that if $\widetilde{\mathcal{A}}(m,n)$ is reducible and $\widetilde{\mathcal{B}}(m,n)$ admits a nonuniform (h, k, μ, ν)-dichotomy, then $\widetilde{\mathcal{A}}(m,n)$ also admits a nonuniform (h, k, μ, ν)-dichotomy.

In Theorem 2.1, we note that a sequence of linear operators admit a nonuniform (h, k, μ, ν)-dichotomy if the Lyapunov exponents are negative in E while all Lyapunov exponents are positive in F. This is a rather weaker assumptions. It also shows that the nonuniform (h, k, μ, ν)-dichotomy should exist widely in the sequence of linear operators and occur naturally.

3. Linear perturbations: roughness

In this section, we consider the roughness or robustness problem for difference equations defined by a sequence of linear operators in a Banach space, or equivalently for a nonautonomous dynamics with discrete time. The principal aim is to show that the (h, k, μ, ν)-dichotomy defined in Section 2 persists under sufficiently small linear perturbations of the original dynamics. In particular, we establish parameter dependence of robustness or roughness of the nonuniform (h, k, μ, ν)-dichotomy in a Banach space X and show that the stable and unstable subspaces of nonuniform (h, k, μ, ν)-dichotomies for the linear perturbed system are Lipschitz continuous in the parameters.

Let $Y = (Y, |\cdot|)$ be an open subset of a Banach space (the parameter space) and consider the nonautonomous dynamics with discrete time

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{Z} \tag{3.1}$$

and the linear perturbed system with parameters

$$x_{m+1} = (A_m + B_m(\lambda))x_m, \tag{3.2}$$

where $B_m : Y \to \mathcal{B}(X)$ are invertible. For each $\lambda \in Y$, define

$$\widehat{\mathcal{A}}_\lambda(m, n) = \begin{cases} (A_{m-1} + B_{m-1}(\lambda)) \cdots (A_n + B_n(\lambda)) & \text{if } m > n, \\ \text{id} & \text{if } m = n, \\ (A_m + B_m(\lambda))^{-1} \cdots (A_{n-1} + B_{n-1}(\lambda))^{-1} & \text{if } m < n. \end{cases}$$

Theorem 3.1. *Assume that*

(a$_1$) $\{A_m\}_{m\in\mathbb{Z}}$ *admits a nonuniform (h, k, μ, ν)-dichotomy;*

(a$_2$) *there exist positive constants $c > 0$ and $\omega > 1$ such that, for any $\lambda, \lambda_1, \lambda_2 \in Y$,*

$$\|B_m(\lambda)\| \le c \min\{(h_{m+1}/h_m)^a \mu_{|m+1|}^{-\omega-\varepsilon}, \nu_{|m+1|}^{-\omega-\varepsilon}\},$$
$$\|B_m(\lambda_1) - B_m(\lambda_2)\| \le c|\lambda_1 - \lambda_2| \cdot \min\{(h_{m+1}/h_m)^a \mu_{|m+1|}^{-\omega-\varepsilon}, \nu_{|m+1|}^{-\omega-\varepsilon}\};$$

(a$_3$) $\lim\limits_{m\to\infty} k_m^{-b} \nu_{|m|}^{\varepsilon} = 0$ *and* $\lim\limits_{m\to-\infty} h_m^{-a} \mu_{|m|}^{\varepsilon} = 0$*;*

(a$_4$) *there are positive constants N_1 and N_2 such that for each $m \in \mathbb{Z}$*

$$\sum_{\tau=-\infty}^{m-1} \mu_{|\tau+1|}^{-\omega} \nu_{|m|}^{\varepsilon} \le N_1, \qquad \sum_{\tau=m}^{\infty} \mu_{|m|}^{\varepsilon} \nu_{|\tau+1|}^{-\omega} \le N_2.$$

If

$$c < [K(2K+1)(N_1+N_2)]^{-1}, \tag{3.3}$$

then the sequence of linear operators $\{A_m + B_m(\lambda)\}_{m\in\mathbb{Z}}$ *also admits a nonuniform* (h,k,μ,ν)*-dichotomy, i.e., for each* $\lambda \in Y$*, there exist projections* $\widehat{P}_n(\lambda)$ *such that*

$$\widehat{P}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n) = \widehat{\mathcal{A}}_\lambda(m,n)\widehat{P}_n(\lambda) \tag{3.4}$$

and

$$\begin{aligned}
\|\widehat{\mathcal{A}}_\lambda(m,n)\widehat{P}_n(\lambda)\| &\le \frac{K\widehat{K}}{1-2K\widehat{K}c(N_1+N_2)}(h_m/h_n)^a \mu_{|n|}^{\varepsilon}(\mu_{|n|}^{\varepsilon}+\nu_{|n|}^{\varepsilon}), m \ge n,\\
\|\widehat{\mathcal{A}}_\lambda(m,n)\widehat{Q}_n(\lambda)\| &\le \frac{K\widehat{K}}{1-2K\widehat{K}c(N_1+N_2)}(k_n/k_m)^{-b} \nu_{|n|}^{\varepsilon}(\mu_{|n|}^{\varepsilon}+\nu_{|n|}^{\varepsilon}), n \ge m,
\end{aligned} \tag{3.5}$$

where $\widehat{Q}_n(\lambda) = \mathrm{id} - \widehat{P}_n(\lambda)$ *are the complementary projections of* $\widehat{P}_n(\lambda)$ *and*

$$\widehat{K} = K/(1-Kc(N_1+N_2)). \tag{3.6}$$

Moreover, if Y *is finite-dimensional, then the stable subspace* $\widehat{P}_n(\lambda)(X)$ *and the unstable subspace* $\widehat{Q}_n(\lambda)(X)$ *are Lipschitz continuous in* λ*.*

In the following discussion of this section, we assume that the conditions in Theorem 3.1 are always satisfied and the proof of Theorem 3.1 will be completed in several steps.

For each $n \in \mathbb{Z}$, define

$$\begin{aligned}
\Omega_1 &:= \{U(m,n)_{m\ge n} \subset \mathcal{B}(X) : \|U\|_1 < \infty\},\\
\Omega_2 &:= \{V(m,n)_{n\ge m} \subset \mathcal{B}(X) : \|V\|_2 < \infty\},
\end{aligned}$$

with the norms

$$\begin{aligned}
\|U\|_1 &= \sup\left\{\|U(m,n)\|(h_m/h_n)^{-a}\mu_{|n|}^{-\varepsilon} : m \ge n\right\},\\
\|V\|_2 &= \sup\left\{\|V(m,n)\|(k_n/k_m)^{b}\nu_{|n|}^{-\varepsilon} : m \le n\right\},
\end{aligned}$$

respectively. Then $(\Omega_1, \|\cdot\|_1)$ and $(\Omega_2, \|\cdot\|_2)$ are Banach spaces.

Lemma 3.1. *For each* $\lambda \in Y$ *and* $n \in \mathbb{Z}$*, there exists a unique solution* $U_\lambda \in \Omega_1$ *of* (3.2) *satisfying*

$$\begin{aligned}
U_\lambda(m,n) = \mathcal{A}(m,n)P_n &+ \sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)U(\tau,n)\\
&- \sum_{\tau=m}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)U(\tau,n)
\end{aligned} \tag{3.7}$$

and $U_\lambda(m,\sigma)U_\lambda(\sigma,n) = U_\lambda(m,n)$ *for* $m \ge \sigma \ge n$*. Moreover,* U_λ *is Lipschitz continuous in* λ*.*

Proof. It is trivial to show that $U_\lambda(m,n)_{m\geq n}$ satisfying (3.7) is a solution of (3.2). For each $\lambda \in Y$, define the operator J_1^λ on Ω_1 by

$$\begin{aligned}(J_1^\lambda U)(m,n) = \mathcal{A}(m,n)P_n + \sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)U(\tau,n) \\ - \sum_{\tau=m}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)U(\tau,n).\end{aligned}$$

We will show that J_1^λ has a unique fixed point in Ω_1. In fact, for $m \geq n$, one has

$$\begin{aligned}A_\lambda^1 := &\sum_{\tau=n}^{m-1}\|\mathcal{A}(m,\tau+1)P_{\tau+1}\|\|B_\tau(\lambda)\|\|U(\tau,n)\| \\ &+ \sum_{\tau=m}^{\infty}\|\mathcal{A}(m,\tau+1)Q_{\tau+1}\|\|B_\tau(\lambda)\|\|U(\tau,n)\| \\ &\leq Kc(h_m/h_n)^a\mu_{|n|}^\varepsilon\sum_{\tau=n}^{m-1}\mu_{|\tau+1|}^{-\omega}\|U\|_1 + Kc(h_m/h_n)^a\mu_{|n|}^\varepsilon\sum_{\tau=m}^{\infty}\nu_{|\tau+1|}^{-\omega}\|U\|_1 \\ &\leq Kc(N_1+N_2)(h_m/h_n)^a\mu_{|n|}^\varepsilon\|U\|_1.\end{aligned}$$

Then,

$$\begin{aligned}\|(J_1^\lambda U)(m,n)\| \leq K(h_m/h_n)^a\mu_{|n|}^\varepsilon + A_\lambda^1 \leq K(h_m/h_n)^a\mu_{|n|}^\varepsilon \\ + Kc(N_1+N_2)(h_m/h_n)^a\mu_{|n|}^\varepsilon\|U\|_1\end{aligned}$$

and

$$\|J_1^\lambda U\|_1 \leq K + Kc(N_1+N_2)\|U\|_1 < \infty. \tag{3.8}$$

Hence, $J_1^\lambda U$ is well-defined and $J_1^\lambda : \Omega_1 \to \Omega_1$. Moreover, for each $\lambda \in Y$, $U_1, U_2 \in \Omega_1$, and $m \geq n$, define

$$A_\lambda^2 := \sum_{\tau=n}^{m-1}\|\mathcal{A}(m,\tau+1)P_{\tau+1}\|\|B_\tau(\lambda)\|\|U_1(\tau,n) - U_2(\tau,n)\|$$

and

$$A_\lambda^3 := \sum_{\tau=m}^{\infty}\|\mathcal{A}(m,\tau+1)Q_{\tau+1}\|\|B_\tau(\lambda)\|\|U_1(\tau,n) - U_2(\tau,n)\|.$$

Then

$$A_\lambda^2 + A_\lambda^3 \leq Kc(N_1+N_2)(h_m/h_n)^a\mu_{|n|}^\varepsilon\|U_1 - U_2\|_1$$

Whence,

$$\begin{aligned}\|(J_1^\lambda U_1)(m,n) - (J_1^\lambda U_2)(m,n)\| &\leq A_\lambda^2 + A_\lambda^3 \\ &\leq Kc(N_1+N_2)(h_m/h_n)^a\mu_{|n|}^\varepsilon\|U_1 - U_2\|_1\end{aligned}$$

and

$$\|J_1^\lambda U_1 - J_1^\lambda U_2\|_1 \leq Kc(N_1+N_2)\|U_1 - U_2\|_1.$$

If (3.3) holds, then the operator J_1^λ is a contraction and there exists a unique $U_\lambda \in \Omega_1$ such that $J_1^\lambda U_\lambda = U_\lambda$. Therefore, (3.7) holds.

By (3.7), one has

$$\begin{aligned}
U_\lambda(m,\sigma)U_\lambda(\sigma,n) = \mathcal{A}(m,n)P_n &+ \sum_{\tau=n}^{\sigma-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)U_\lambda(\tau,n) \\
&+ \sum_{\tau=\sigma}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)U_\lambda(\tau,\sigma)U_\lambda(\sigma,n) \\
&- \sum_{\tau=m}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)U_\lambda(\tau,\sigma)U_\lambda(\sigma,n).
\end{aligned}$$

Let $L_\lambda(m,\sigma) = U_\lambda(m,\sigma)U_\lambda(\sigma,n) - U_\lambda(m,n)$ for $m \ge \sigma \ge n$. For $l \in \Omega_1^\sigma$ (here Ω_1^σ is Ω_1 with n replaced by σ) , $m \ge \sigma$, and $\lambda \in Y$, define the operator H_1^λ by

$$\begin{aligned}
(H_1^\lambda l)(m,\sigma) = &\sum_{\tau=\sigma}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)l(\tau,\sigma) \\
&- \sum_{\tau=m}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)l(\tau,\sigma).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|(H_1^\lambda l)(m,\sigma)\| &\le Kc(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \sum_{\tau=n}^{m-1} \mu_{|\tau+1|}^{-\omega} \|l\|_1 \\
&\quad + Kc(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \sum_{\tau=m}^{\infty} \nu_{|\tau+1|}^{-\omega} \|l\|_1 \\
&\le Kc(N_1+N_2)(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \|l\|_1
\end{aligned}$$

and

$$\begin{aligned}
\|(H_1^\lambda l_1)(m,\sigma) - (H_1^\lambda l_2)(m,\sigma)\| &\le Kc(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \sum_{\tau=n}^{m-1} \mu_{|\tau+1|}^{-\omega} \|l_1 - l_2\|_1 \\
&\quad + Kc(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \sum_{\tau=m}^{\infty} \nu_{|\tau+1|}^{-\omega} \|l_1 - l_2\|_1 \\
&\le Kc(N_1+N_2)(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \|l_1 - l_2\|_1
\end{aligned}$$

for $l, l_1, l_2 \in \Omega_1^\sigma$. Then

$$\|H_1^\lambda l\|_1 \le Kc(N_1+N_2)\|l\|_1 < \infty$$

and

$$\|H_1^\lambda l_1 - H_1^\lambda l_2\|_1 \le Kc(N_1+N_2)\|l_1 - l_2\|_1.$$

Therefore, H_1^λ is well-defined, $H_1^\lambda(\Omega_1^\sigma) \subset \Omega_1^\sigma$, and there exists a unique $l_\lambda \in \Omega_1^\sigma$ such that $H_1^\lambda l_\lambda = l_\lambda$. Moreover, it is not difficult to show that $0 \in \Omega_1^\sigma$ and $H_1^\lambda 0 = 0$. On the other hand, it is clear that $H_1^\lambda L_\lambda = L_\lambda$. Whence $L_\lambda = l_\lambda = 0$.

It is time to show that U_λ is Lipschitz continuous in λ. It is clear that, for any $\lambda_1, \lambda_2 \in Y$, there exist bounded solutions $U_{\lambda_1}, U_{\lambda_2} \in \Omega_1$ satisfying (3.7). By (a_2) and (3.8), we have

$$\begin{aligned} A^4(\tau) :&= \|B_\tau(\lambda_1)U_{\lambda_1}(\tau,n) - B_\tau(\lambda_2)U_{\lambda_2}(\tau,n)\| \\ &\le \|B_\tau(\lambda_1)U_{\lambda_1}(\tau,n) - B_\tau(\lambda_1)U_{\lambda_2}(\tau,n)\| \\ &\quad + \|B_\tau(\lambda_1)U_{\lambda_2}(\tau,n) - B_\tau(\lambda_2)U_{\lambda_2}(\tau,n)\| \\ &\le c(h_{\tau+1}/h_n)^a \mu_{|\tau+1|}^{-\omega-\varepsilon} \mu_{|n|}^{\varepsilon} (\|U_{\lambda_1} - U_{\lambda_2}\|_1 + \widehat{K}|\lambda_1 - \lambda_2|) \end{aligned}$$

for any $\tau \ge n$. It follows from (3.7) that

$$\begin{aligned} &\|U_{\lambda_1}(m,n) - U_{\lambda_2}(m,n)\| \\ &\le \sum_{\tau=n}^{m-1} \|\mathcal{A}(m,\tau+1)P_{\tau+1}\| A^4(\tau) + \sum_{\tau=m}^{\infty} \|\mathcal{A}(m,\tau+1)Q_{\tau+1}\| A^4(\tau) \\ &\le Kc(h_m/h_n)^a \mu_{|n|}^{\varepsilon} \left(\sum_{\tau=n}^{m-1} \mu_{|\tau+1|}^{-\omega} + \sum_{\tau=m}^{\infty} \nu_{|\tau+1|}^{-\omega} \right) (\|U_{\lambda_1} - U_{\lambda_2}\|_1 + \widehat{K}|\lambda_1 - \lambda_2|) \\ &\le Kc(N_1+N_2)(h_m/h_n)^a \mu_{|n|}^{\varepsilon} (\|U_{\lambda_1} - U_{\lambda_2}\|_1 + \widehat{K}|\lambda_1 - \lambda_2|). \end{aligned}$$

Then

$$\|U_{\lambda_1} - U_{\lambda_2}\|_1 \le [\widehat{K}Kc(N_1+N_2)/(1 - Kc(N_1+N_2))] \cdot |\lambda_1 - \lambda_2|.$$

The proof is complete. ∎

Lemma 3.2. *For $\lambda \in Y$ and $n \in \mathbb{Z}$, there exists a unique solution $V_\lambda \in \Omega_2$ of (3.2) satisfying*

$$\begin{aligned} V_\lambda(m,n) = \mathcal{A}(m,n)Q_n &+ \sum_{\tau=-\infty}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,n) \\ &- \sum_{\tau=m}^{n-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,n) \end{aligned} \tag{3.9}$$

and $V_\lambda(m,\sigma)V_\lambda(\sigma,n) = V_\lambda(m,n)$ for $n \ge \sigma \ge m$. Moreover, V_λ is Lipschitz continuous in λ.

Proof. It is obvious that $V_\lambda(m,n)_{n\ge m}$ satisfying (3.9) is a solution of (3.2). For each $\lambda \in Y$, define the operator J_2^λ in Ω_2 by

$$\begin{aligned} (J_2^\lambda V)(m,n) = \mathcal{A}(m,n)Q_n &+ \sum_{\tau=-\infty}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)V(\tau,n) \\ &- \sum_{\tau=m}^{n-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)V(\tau,n). \end{aligned}$$

It follows from (2.1) that

$$
\begin{aligned}
A_\lambda^5 := &\sum_{\tau=-\infty}^{m-1} \|\mathcal{A}(m,\tau+1)P_{\tau+1}\|\|B_\tau(\lambda)\|\|V(\tau,n)\| \\
&+ \sum_{\tau=m}^{n-1} \|\mathcal{A}(m,\tau+1)Q_{\tau+1}\|\|B_\tau(\lambda)\|\|V(\tau,n)\| \\
\le\, & Kc(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon} \sum_{\tau=-\infty}^{m-1} \mu_{|\tau+1|}^{-\omega}\|V\|_2 \\
&+ Kc(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon} \sum_{\tau=m}^{n-1} \nu_{|\tau+1|}^{-\omega}\|V\|_2 \\
\le\, & Kc(N_1+N_2)(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|V\|_2
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
\|(J_2^\lambda V)(m,n)\| &\le K(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon} + A_\lambda^5 \\
&\le K(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon} + Kc(N_1+N_2)(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|V\|_2.
\end{aligned}
$$

Then

$$
\|J_2^\lambda V\|_2 \le K + Kc(N_1+N_2)\|V\|_2 < \infty \tag{3.11}
$$

and $J_2^\lambda : \Omega_2 \to \Omega_2$ is well-defined. Proceeding in a manner similar to those in (3.10), one has

$$
\|J_2^\lambda V_1 - J_2^\lambda V_2\|_2 \le Kc(N_1+N_2)\|V_1 - V_2\|_2.
$$

The operator J_2^λ is a contraction due to (3.3) and then there exists a unique $V_\lambda \in \Omega_2$ such that $J_2^\lambda V_\lambda = V_\lambda$. Hence (3.9) holds.

From (3.9), it follows that

$$
\begin{aligned}
V_\lambda(m,\sigma)V_\lambda(\sigma,n) = \mathcal{A}(m,n)Q_n &- \sum_{\tau=\sigma}^{n-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,n) \\
&+ \sum_{\tau=1}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,\sigma)V_\lambda(\sigma,n) \\
&- \sum_{\tau=m}^{\sigma-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,\sigma)V_\lambda(\sigma,n).
\end{aligned}
$$

For a fixed $\sigma \in \mathbb{Z}$, let $L_\lambda^*(m,\sigma) = V_\lambda(m,\sigma)V_\lambda(\sigma,n) - V_\lambda(m,n)$ for $n \ge \sigma \ge m$. Consider the operator H_2^λ defined by

$$
\begin{aligned}
(H_2^\lambda l^*)(m,\sigma) = &\sum_{\tau=1}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)l^*(\tau,\sigma) \\
&- \sum_{\tau=m}^{\sigma-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)l^*(\tau,\sigma)
\end{aligned}
$$

for $\lambda \in Y$, $l^* \in \Omega_2^\sigma$, and $m \geq \sigma$, where Ω_2^σ is obtained from Ω_2 by replacing n with σ. It is not difficult to show that $H_2^\lambda L^* = L^*$, $\|H_2^\lambda l^*\|_2 \leq Kc(N_1+N_2)\|l^*\|_2$ and

$$\|H_2^\lambda l_1^* - H_2^\lambda l_2^*\|_2 \leq Kc(N_1+N_2)\|l_1^* - l_2^*\|_2$$

for $l^*, l_1^*, l_2^* \in \Omega_2^\sigma$. Then there exists a unique $l^* \in \Omega_2^\sigma$ such that $H_2^\lambda l_\lambda^* = l_\lambda^*$ and $l_\lambda^* = L_\lambda^*$. Moreover, $0 \in \Omega_2^\sigma$ also satisfies this identity and $H_2^\lambda 0 = 0$, which then implies that $L_\lambda^* = l_\lambda^* = 0$.

Next we show that V_λ is Lipschitz continuous in λ. For any $\lambda_1, \lambda_2 \in Y$, there exist bounded solutions $V_{\lambda_1}, V_{\lambda_2} \in \Omega_1$ satisfying (3.9). It follows from (a$_2$) and (3.11) that

$$\begin{aligned}
A^6(\tau) : &= \|B_\tau(\lambda_1)V_{\lambda_1}(\tau,n) - B_\tau(\lambda_2)V_{\lambda_2}(\tau,n)\| \\
&\leq \|B_\tau(\lambda_1)V_{\lambda_1}(\tau,n) - B_\tau(\lambda_1)V_{\lambda_2}(\tau,n)\| \\
&\quad + \|B_\tau(\lambda_1)V_{\lambda_2}(\tau,n) - B_\tau(\lambda_2)V_{\lambda_2}(\tau,n)\| \\
&\leq c(k_\tau/k_n)^a \nu_{|\tau+1|}^{-\omega-\varepsilon}\nu_{|n|}^{\varepsilon}(\|V_{\lambda_1} - V_{\lambda_2}\|_2 + \widehat{K}|\lambda_1 - \lambda_2|)
\end{aligned}$$

for any $\tau \geq n$. By (3.9), one has

$$\begin{aligned}
&\|V_{\lambda_1}(m,n) - V_{\lambda_2}(m,n)\| \\
&\leq \sum_{\tau=1}^{m-1} \|\mathcal{A}(m,\tau+1)P_{\tau+1}\|A^6(\tau) + \sum_{\tau=m}^{n-1} \|\mathcal{A}(m,\tau+1)Q_{\tau+1}\|A^6(\tau) \\
&\leq Kc(k_m/k_n)^a \nu_{|n|}^{\varepsilon}\left(\sum_{\tau=1}^{m-1}\mu_{|\tau+1|}^{-\omega} + \sum_{\tau=m}^{n-1}\nu_{\tau+1}^{-\omega}\right)(\|V_{\lambda_1} - V_{\lambda_2}\|_2 + \widehat{K}|\lambda_1 - \lambda_2|) \\
&\leq Kc(N_1+N_2)(k_m/k_n)^a \nu_{|n|}^{\varepsilon}(\|V_{\lambda_1} - V_{\lambda_2}\|_2 + \widehat{K}|\lambda_1 - \lambda_2|).
\end{aligned}$$

Then

$$\|V_{\lambda_1} - V_{\lambda_2}\|_2 \leq [\widehat{K}Kc(N_1+N_2)/(1 - Kc(N_1+N_2))] \cdot |\lambda_1 - \lambda_2|.$$

The proof is complete. ∎

For $\lambda \in Y$ and $m \in \mathbb{Z}$, define

$$\widetilde{P}_m(\lambda) = \widehat{\mathcal{A}}_\lambda(m,0)U_\lambda(0,0)\widehat{\mathcal{A}}_\lambda(0,m), \quad \widetilde{Q}_m(\lambda) = \widehat{\mathcal{A}}_\lambda(m,0)V_\lambda(0,0)\widehat{\mathcal{A}}_\lambda(0,m).$$

Then $U_\lambda(m,0)P_0 = U_\lambda(m,0)$ since $\widehat{U}_\lambda(m,0) = U_\lambda(m,0)P_0$ satisfies (3.7) with $n = 0$ and $V_\lambda(m,0)Q_0 = V_\lambda(m,0)$ since $\widehat{V}_\lambda(m,0) = V_\lambda(m,0)Q_0$ satisfies (3.9) with $n = 0$. For $\lambda \in Y$, from Lemmas 3.1, 3.2 and

$$\begin{aligned}
\widetilde{P}_0(\lambda) &= U_\lambda(0,0) = P_0 - \sum_{\tau=0}^{\infty}\mathcal{A}(0,\tau+1)Q_{\tau+1}B_\tau(\lambda)U_\lambda(\tau,0), \\
\widetilde{Q}_0(\lambda) &= V_\lambda(0,0) = Q_0 + \sum_{\tau=-\infty}^{-1}\mathcal{A}(0,\tau+1)P_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,0),
\end{aligned} \tag{3.12}$$

it follows that

(b$_1$) $\widetilde{P}_m(\lambda)$ and $\widetilde{Q}_m(\lambda)$ are projections for $m \in \mathbb{Z}$;

(b$_2$) $\widetilde{P}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n) = \widehat{\mathcal{A}}_\lambda(m,n)\widetilde{P}_n(\lambda), \widetilde{Q}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n) = \widehat{\mathcal{A}}_\lambda(m,n)\widetilde{Q}_n(\lambda)$ for $m,\, n \in \mathbb{Z}$;

(b$_3$) $P_0\widetilde{P}_0(\lambda) = P_0, Q_0\widetilde{Q}_0(\lambda) = Q_0, Q_0(\mathrm{id} - \widetilde{P}_0(\lambda)) = \mathrm{id} - \widetilde{P}_0(\lambda), P_0(\mathrm{id} - \widetilde{Q}_0(\lambda)) = \mathrm{id} - \widetilde{Q}_0(\lambda)$;

(b$_4$) $\widetilde{P}_0(\lambda)P_0 = \widetilde{P}_0(\lambda), \widetilde{Q}_0(\lambda)Q_0 = \widetilde{Q}_0(\lambda)$.

Lemma 3.3. *For $\lambda \in Y$, one has*

$$\|\widehat{\mathcal{A}}_\lambda(m,n)|\,\mathrm{Im}\,\widetilde{P}_n(\lambda)\| \le \widehat{K}(h_m/h_n)^a \mu^\varepsilon_{|n|}, m \ge n,$$
$$\|\widehat{\mathcal{A}}_\lambda(m,n)|\,\mathrm{Im}\,\widetilde{Q}_n(\lambda)\| \le \widehat{K}(k_n/k_m)^{-b} \nu^\varepsilon_{|n|}, m \le n.$$

Proof. By the variation-of-constants formula, for $\lambda \in Y$ and $m \in \mathbb{Z}$, if $(z_m^\lambda)_{m\ge n}$ is a solution of (3.2), then $z_m^\lambda = P_m z_m^\lambda + Q_m z_m^\lambda$, where

$$P_m z_m^\lambda = \mathcal{A}(m,n)P_n z_n^\lambda + \sum_{\tau=n}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)z_\tau^\lambda,$$
$$Q_m z_m^\lambda = \mathcal{A}(m,n)Q_n z_n^\lambda + \sum_{\tau=n}^{m-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda. \tag{3.13}$$

Our strategy here is to show that, if $(z_m^\lambda)_{m\ge n}$ is bounded, then

$$\begin{aligned} z_m^\lambda = {} & \mathcal{A}(m,n)P_n z_n^\lambda + \sum_{\tau=n}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)z_\tau^\lambda \\ & - \sum_{\tau=m}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda, \quad m \ge n. \end{aligned} \tag{3.14}$$

By (3.13), we have

$$Q_n z_n^\lambda = \mathcal{A}(n,m)Q_m z_m^\lambda - \sum_{\tau=n}^{m-1} \mathcal{A}(n,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda. \tag{3.15}$$

Moreover, $\|\mathcal{A}(n,m)Q_m\| \le K(k_m/k_n)^{-b}\nu^\varepsilon_{|m|}$ and

$$\begin{aligned} \sum_{\tau=n}^{\infty} \|\mathcal{A}(n,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda\| &\le Kc\sum_{\tau=n}^{\infty} \nu^{-\omega}_{|\tau+1|} \sup_{\tau\ge n}\|z_\tau^\lambda\| \\ &\le KcN_2 \sup_{\tau\ge n}\|z_\tau^\lambda\| < \infty. \end{aligned}$$

Then, $Q_n z_n = -\sum\limits_{\tau=n}^{\infty} \mathcal{A}(n,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda$ by letting $m \to \infty$ in (3.15). Hence,

$$\begin{aligned} Q_m z_m^\lambda &= -\sum_{\tau=n}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda + \sum_{\tau=n}^{m-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda \\ &= -\sum_{\tau=m}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda, \end{aligned}$$

which proves (3.14).

Given $\xi \in X$, for $\lambda \in Y$, consider the solution $z_m^\lambda = \widehat{\mathcal{A}}_\lambda(m,n)\widetilde{P}_n(\lambda)\xi$ of (3.2) for $m \geq n$. By the fact that $\widehat{\mathcal{A}}_\lambda(m,0)U_\lambda(0,0)$ and $U_\lambda(m,0)$ are solutions of (3.2), which coincide for $m=0$, we have

$$z_m^\lambda := \widehat{\mathcal{A}}_\lambda(m,0)U_\lambda(0,0)\widehat{\mathcal{A}}_\lambda(0,n)\xi = U_\lambda(m,0)\widehat{\mathcal{A}}_\lambda(0,n)\xi.$$

Then $(z_m^\lambda)_{m\geq n}$ is a bounded solution of (3.2) with the initial value $z_n^\lambda = \widetilde{P}_n(\lambda)\xi$ since $U_\lambda(m,0)$ is bounded for $m \in \mathbb{Z}$. From (3.14), for $m \geq n$, it follows that

$$\begin{aligned}
\widetilde{P}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n)\xi = {} & \mathcal{A}(m,n)P_n\widetilde{P}_n(\lambda)\xi \\
& + \sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)\widetilde{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi \\
& - \sum_{\tau=m}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)\widetilde{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi.
\end{aligned}$$

Moreover,

$$\begin{aligned}
A_\lambda^7 : &= \sum_{\tau=n}^{m-1}\|\mathcal{A}(m,\tau+1)P_{\tau+1}\|\|B_\tau(\lambda)\|\|\widetilde{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi\| \\
&\leq Kc\sum_{\tau=n}^{m-1}(h_m/h_\tau)^a\mu_{|\tau+1|}^{-\omega}\|\widetilde{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\|\|\widetilde{P}_n(\lambda)\xi\| \\
&\leq Kc(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}(\lambda)\widehat{\mathcal{A}}_\lambda\|_1\|\widetilde{P}_n(\lambda)\xi\|\sum_{\tau=n}^{m-1}\mu_{|\tau+1|}^{-\omega} \\
&\leq Kc(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}(\lambda)\widehat{\mathcal{A}}_\lambda\|_1\|\widetilde{P}_n(\lambda)\xi\|N_1
\end{aligned}$$

and

$$\begin{aligned}
A_\lambda^8 : &= \sum_{\tau=m}^{\infty}\|\mathcal{A}(m,\tau+1)Q_{\tau+1}\|\|B_\tau(\lambda)\|\|\widetilde{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi\| \\
&\leq Kc\sum_{\tau=m}^{\infty}(k_{\tau+1}/k_m)^{-b}\nu_{|\tau+1|}^{-\omega}\|\widetilde{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\|\|\widetilde{P}_n(\lambda)\xi\| \\
&\leq Kc(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}(\lambda)\widehat{\mathcal{A}}_\lambda\|_1\|\widetilde{P}_n(\lambda)\xi\|\sum_{\tau=m}^{\infty}\nu_{|\tau+1|}^{-\omega} \\
&\leq Kc(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}(\lambda)\widehat{\mathcal{A}}_\lambda\|_1\|\widetilde{P}_n(\lambda)\xi\|N_2.
\end{aligned}$$

Then

$$\begin{aligned}
\|\widetilde{P}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n)\xi\| &\leq K(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}_n(\lambda)\xi\| + A_\lambda^7 + A_\lambda^8 \\
&\leq K(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}_n(\lambda)\xi\| \\
&\quad + Kc(N_1+N_2)(h_m/h_n)^a\mu_{|n|}^{\varepsilon}\|\widetilde{P}(\lambda)\widehat{\mathcal{A}}_\lambda\|_1\|\widetilde{P}_n(\lambda)\xi\|
\end{aligned}$$

and $\|\widetilde{P}(\lambda)\widehat{\mathcal{A}}_\lambda\|_1 \leq \widehat{K}$. Therefore, the first inequality holds.

By carrying out similar arguments, we claim that, for each $\lambda \in Y$, if $(z_m^\lambda)_{m\leq n}$ is a bounded solution of (3.2) and $\lim\limits_{m\to-\infty} h_m^{-a}\mu_{|m|}^{\varepsilon} = 0$, then

$$\begin{aligned} z_m^\lambda = \mathcal{A}(m,n)Q_n z_n^\lambda + &\sum_{\tau=-\infty}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)z_\tau^\lambda \\ &- \sum_{\tau=m}^{n-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)z_\tau^\lambda. \end{aligned} \tag{3.16}$$

Given $\xi \in X$ and $\lambda \in Y$, one has

$$z_m^\lambda := \widehat{\mathcal{A}}_\lambda(m,n)\widetilde{Q}_n(\lambda)\xi = V_\lambda(m,0)\widehat{\mathcal{A}}_\lambda(0,n)\xi,\ m \leq n$$

and $(z_m^\lambda)_{m\leq n}$ is a bounded solution of (3.2) with $z_n^\lambda = \widetilde{Q}_n(\lambda)\xi$. Then, by (3.16),

$$\begin{aligned} \widetilde{Q}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n)\xi = \mathcal{A}(m,n)&Q_n\widetilde{Q}_n(\lambda)\xi \\ &+ \sum_{\tau=-\infty}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)\widetilde{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi \\ &- \sum_{\tau=m}^{n-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)\widetilde{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi. \end{aligned}$$

Note that

$$\begin{aligned} A_\lambda^9 :&= \sum_{\tau=-\infty}^{m-1} \|\mathcal{A}(m,\tau+1)P_{\tau+1}\|\|B_\tau(\lambda)\|\|\widetilde{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi\| \\ &\leq Kc\sum_{\tau=-\infty}^{m-1} (h_m/h_\tau)^a\mu_{|\tau+1|}^{-\omega}\|\widetilde{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\|\|\widetilde{Q}_n(\lambda)\xi\| \\ &\leq Kc(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2\|\widetilde{Q}_n(\lambda)\xi\| \sum_{\tau=-\infty}^{m-1} \mu_{|\tau+1|}^{-\omega} \\ &\leq Kc(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2\|\widetilde{Q}_n(\lambda)\xi\| N_1 \end{aligned}$$

and

$$\begin{aligned} A_\lambda^{10} :&= \sum_{\tau=m}^{n-1} \|\mathcal{A}(m,\tau+1)Q_{\tau+1}\|\|B_\tau(\lambda)\|\|\widetilde{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi\| \\ &\leq Kc\sum_{\tau=m}^{n-1} (k_{\tau+1}/k_m)^{-b}\nu_{|\tau+1|}^{-\omega}\|\widetilde{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\|\|\widetilde{Q}_n(\lambda)\xi\| \\ &\leq Kc(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2\|\widetilde{Q}_n(\lambda)\xi\| \sum_{\tau=m}^{n-1} \nu_{|\tau+1|}^{-\omega} \\ &\leq Kc(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2\|\widetilde{Q}_n(\lambda)\xi\| N_2, \end{aligned}$$

then

$$\begin{aligned}\|\widetilde{Q}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n)\xi\| &\le K(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}_n(\lambda)\xi\| + A_\lambda^9 + A_\lambda^{10}\\ &\le K(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}_n(\lambda)\xi\|\\ &\quad + Kc(N_1+N_2)(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2\|\widetilde{Q}_n(\lambda)\xi\|\end{aligned}$$

and

$$\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2 \le K + Kc(N_1+N_2)\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2,$$

i.e., $\|\widetilde{Q}(\lambda)\widehat{\mathcal{A}}_\lambda\|_2 \le \widehat{K}$, which yields the second inequality. ∎

Next, we construct the projections $\widehat{P}_m(\lambda)$ for $\lambda \in Y$.

Lemma 3.4. *For $\lambda \in Y$, $S_0(\lambda) = \widetilde{P}_0(\lambda) + \widetilde{Q}_0(\lambda)$ is invertible.*

Proof. By(3.12), (b_3), and (b_4), one has

$$\widetilde{P}_0(\lambda) + \widetilde{Q}_0(\lambda) - \mathrm{id} = Q_0\widetilde{P}_0(\lambda) + P_0\widetilde{Q}_0(\lambda), \tag{3.17}$$

where

$$\begin{aligned}P_0\widetilde{Q}_0(\lambda) &= P_0V_\lambda(0,0) = \sum_{\tau=-\infty}^{-1}\mathcal{A}(0,\tau+1)P_{\tau+1}B_\tau(\lambda)V_\lambda(\tau,0),\\ Q_0\widetilde{P}_0(\lambda) &= Q_0U_\lambda(0,0) = -\sum_{\tau=0}^{\infty}\mathcal{A}(0,\tau+1)Q_{\tau+1}B_\tau(\lambda)U_\lambda(\tau,0).\end{aligned}$$

By (3.6), (3.8) and (3.11), for $\lambda \in Y$,

$$\|U_\lambda(m,n)\| \le \widehat{K}(h_m/h_n)^a\mu_{|n|}^{\varepsilon},\ m \ge n \tag{3.18}$$

and

$$\|V_\lambda(m,n)\| \le \widehat{K}(k_n/k_m)^{-b}\nu_{|n|}^{\varepsilon},\ m \le n. \tag{3.19}$$

From (3.17)-(3.19), it follows that

$$\begin{aligned}A_\lambda^{11} :&= \sum_{\tau=-\infty}^{-1}\|\mathcal{A}(0,\tau+1)P_{\tau+1}\|\|B_\tau(\lambda)\|\|V_\lambda(\tau,0)\|\\ &\le K\widehat{K}c\sum_{\tau=-\infty}^{-1}(h_0/h_\tau)^a(k_0/k_\tau)^{-b}\mu_{|\tau+1|}^{-\omega}\nu_0^{\varepsilon}\\ &\le K\widehat{K}c\sum_{\tau=-\infty}^{-1}\mu_{|\tau+1|}^{-\omega} \le K\widehat{K}cN_1\end{aligned}$$

and

$$\begin{aligned}A_\lambda^{12} :&= \sum_{\tau=0}^{\infty}\|\mathcal{A}(0,\tau+1)Q_{\tau+1}\|\|B_\tau(\lambda)\|\|U_\lambda(\tau,0)\|\\ &\le K\widehat{K}c\sum_{\tau=0}^{\infty}(k_{\tau+1}/k_0)^{-b}(h_\tau/h_0)^a\nu_{|\tau+1|}^{-\omega}\mu_0 \le K\widehat{K}c\sum_{\tau=0}^{\infty}\nu_{|\tau+1|}^{-\omega} \le K\widehat{K}cN_2.\end{aligned}$$

Then

$$\|\widetilde{P}_0(\lambda)+\widetilde{Q}_0(\lambda)-\mathrm{id}\,\|\le A_\lambda^{11}+A_\lambda^{12}\le K\widehat{K}c(N_1+N_2)$$

and, by (3.3), $S_0(\lambda)$ is invertible for $\lambda\in Y$. ∎

For $\lambda\in Y$ and $m\in\mathbb{Z}$, set

$$\begin{aligned}\widehat{P}_m(\lambda)&=\widehat{\mathcal{A}}_\lambda(m,0)S_0(\lambda)P_0(\lambda)S_0^{-1}(\lambda)\widehat{\mathcal{A}}_\lambda(0,m),\\ \widehat{Q}_m(\lambda)&=\widehat{\mathcal{A}}_\lambda(m,0)S_0(\lambda)Q_0(\lambda)S_0^{-1}(\lambda)\widehat{\mathcal{A}}_\lambda(0,m).\end{aligned}\tag{3.20}$$

Then $\widehat{P}_m(\lambda)$ and $\widehat{Q}_m(\lambda)$ are projections satisfying (3.4) and $\widehat{P}_m(\lambda)+\widehat{Q}_m(\lambda)=\mathrm{id}$.

Lemma 3.5. *For $\lambda\in Y$, the following claims hold*

$$\begin{aligned}\|\widehat{\mathcal{A}}_\lambda(m,n)\widehat{P}_n(\lambda)\|&\le\widehat{K}(h_m/h_n)^a\mu_{|n|}^\varepsilon\|\widehat{P}_n(\lambda)\|,\quad m\ge n,\\ \|\widehat{\mathcal{A}}_\lambda(m,n)\widehat{Q}_n(\lambda)\|&\le\widehat{K}(k_n/k_m)^{-b}\nu_{|n|}^\varepsilon\|\widehat{Q}_n(\lambda)\|,\quad m\le n.\end{aligned}\tag{3.21}$$

Proof. By (b_4), for $\lambda\in Y$, one has

$$\begin{aligned}S_0(\lambda)P_0&=(\widetilde{P}_0(\lambda)+\widetilde{Q}_0(\lambda))P_0=\widetilde{P}_0(\lambda),\\ S_0(\lambda)Q_0&=(\widetilde{P}_0(\lambda)+\widetilde{Q}_0(\lambda))Q_0=\widetilde{Q}_0(\lambda).\end{aligned}$$

Note that $S_m(\lambda)=\widehat{\mathcal{A}}_\lambda(m,0)S_0(\lambda)\widehat{\mathcal{A}}_\lambda(0,m)$ for $m\in\mathbb{Z}$, then

$$\begin{aligned}\widehat{P}_m(\lambda)S_m(\lambda)&=\widehat{\mathcal{A}}_\lambda(m,0)S_0(\lambda)P_0\widehat{\mathcal{A}}_\lambda(0,m)\\ &=\widehat{\mathcal{A}}_\lambda(m,0)\widetilde{P}_0(\lambda)\widehat{\mathcal{A}}_\lambda(0,m)=\widetilde{P}_m(\lambda).\end{aligned}$$

Similarly, $\widehat{Q}_m(\lambda)S_m(\lambda)=\widetilde{Q}_m(\lambda)$. Whence, $\operatorname{Im}\widehat{P}_m(\lambda)=\operatorname{Im}\widetilde{P}_m(\lambda)$ and $\operatorname{Im}\widehat{Q}_m(\lambda)=\operatorname{Im}\widetilde{Q}_m(\lambda)$ for $\lambda\in Y$ since $S_m(\lambda)$ is invertible. By Lemma 3.3, for $\lambda\in Y$, one has

$$\begin{aligned}\|\widehat{\mathcal{A}}_\lambda(m,n)\widehat{P}_n(\lambda)\|&\le\|\widehat{\mathcal{A}}_\lambda(m,n)|\operatorname{Im}\widetilde{P}_n(\lambda)\|\|\widehat{P}_n(\lambda)\|\\ &\le\widehat{K}(h_m/h_n)^a\mu_{|n|}^\varepsilon\|\widehat{P}_n(\lambda)\|,\quad m\ge n,\\ \|\widehat{\mathcal{A}}_\lambda(m,n)\widehat{Q}_n(\lambda)\|&\le\|\widehat{\mathcal{A}}_\lambda(m,n)|\operatorname{Im}\widetilde{Q}_n(\lambda)\|\|\widehat{Q}_n(\lambda)\|\\ &\le\widehat{K}(k_n/k_m)^{-b}\nu_{|n|}^\varepsilon\|\widehat{Q}_n(\lambda)\|,\quad m\le n.\end{aligned}$$

∎

Lemma 3.6. *For $\lambda\in Y$, the following claims hold*

$$\begin{aligned}\|\widehat{P}_m(\lambda)\|&\le[K/(1-2K\widehat{K}c(N_1+N_2))](\mu_{|m|}^\varepsilon+\nu_{|m|}^\varepsilon),\\ \|\widehat{Q}_m(\lambda)\|&\le[K/(1-2K\widehat{K}c(N_1+N_2))](\mu_{|m|}^\varepsilon+\nu_{|m|}^\varepsilon).\end{aligned}\tag{3.22}$$

Proof. For $\xi \in X$ and $\lambda \in Y$, set

$$z_m^1 = \widehat{\mathcal{A}}_\lambda(m,n)\widehat{P}_n(\lambda)\xi, \quad m \geq n, \qquad z_m^2 = \widehat{\mathcal{A}}_\lambda(m,n)\widehat{Q}_n(\lambda)\xi, \quad m \leq n.$$

Then, by Lemma 3.5, $(z_m^1)_{m\geq n}$ and $(z_m^2)_{m\leq n}$ are bounded solutions of (3.2). By (3.14) and (3.16), one has

$$\begin{aligned}
\widehat{P}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n)\xi &= \mathcal{A}(m,n)P_n\widehat{P}_n(\lambda)\xi \\
&\quad + \sum_{\tau=n}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)\widehat{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi \\
&\quad - \sum_{\tau=m}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)\widehat{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi
\end{aligned}$$

and

$$\begin{aligned}
\widehat{Q}_m(\lambda)\widehat{\mathcal{A}}_\lambda(m,n)\xi &= \mathcal{A}(m,n)Q_n\widehat{Q}_n(\lambda)\xi \\
&\quad + \sum_{\tau=-\infty}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)\widehat{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,n)\xi \\
&\quad - \sum_{\tau=m}^{n-1} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)\widehat{Q}_\tau\widehat{\mathcal{A}}(\tau,n)\xi.
\end{aligned}$$

Taking $m = n$ leads to

$$\begin{aligned}
Q_m\widehat{P}_m(\lambda)\xi &= -\sum_{\tau=m}^{\infty} \mathcal{A}(m,\tau+1)Q_{\tau+1}B_\tau(\lambda)\widehat{P}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,m)\xi, \\
P_m\widehat{Q}_m(\lambda)\xi &= \sum_{\tau=-\infty}^{m-1} \mathcal{A}(m,\tau+1)P_{\tau+1}B_\tau(\lambda)\widehat{Q}_\tau(\lambda)\widehat{\mathcal{A}}_\lambda(\tau,m)\xi.
\end{aligned}$$

By Lemma 3.5,

$$\begin{aligned}
\|Q_m\widehat{P}_m(\lambda)\| &\leq K\widehat{K}c\|\widehat{P}_m(\lambda)\| \sum_{\tau=m}^{\infty} (k_{\tau+1}/k_m)^{-b}(h_\tau/h_m)^a \nu_{|\tau+1|}^{-\omega}\mu_{|m|}^{\varepsilon} \\
&\leq K\widehat{K}cN_2\|\widehat{P}_m(\lambda)\|
\end{aligned}$$

and

$$\begin{aligned}
\|P_m\widehat{Q}_m(\lambda)\| &\leq K\widehat{K}c\|\widehat{Q}_m(\lambda)\| \sum_{\tau=-\infty}^{m-1} (h_m/h_\tau)^a(k_m/k_\tau)^{-b}\mu_{|\tau+1|}^{-\omega}\nu_{|m|}^{\varepsilon} \\
&\leq K\widehat{K}cN_1\|\widehat{Q}_m(\lambda)\|.
\end{aligned}$$

Since $\|P_m\| \le K\mu^{\varepsilon}_{|m|}$ and $\|Q_m\| \le K\nu^{\varepsilon}_{|m|}$, one has

$$
\begin{aligned}
\|\widehat{P}_m(\lambda)\| &\le \|\widehat{P}_m(\lambda) - P_m\| + \|P_m\| \\
&= \|\widehat{P}_m(\lambda) - P_m\widehat{P}_m(\lambda) - P_m + P_m\widehat{P}_m(\lambda)\| + \|P_m\| \\
&= \|Q_m\widehat{P}_m(\lambda) - P_m\widehat{Q}_m(\lambda)\| + \|P_m\| \\
&\le \|Q_m\widehat{P}_m(\lambda)\| + \|P_m\widehat{Q}_m(\lambda)\| + \|P_m\| \\
&\le K\widehat{K}c(N_1+N_2)(\|\widehat{P}_m(\lambda)\| + \|\widehat{Q}_m(\lambda)\|) + K\mu^{\varepsilon}_{|m|}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\widehat{Q}_m(\lambda)\| &\le \|\widehat{Q}_m(\lambda) - Q_m\| + \|Q_m\| = \|\widehat{P}_m(\lambda) - P_m\| + \|Q_m\| \\
&\le K\widehat{K}c(N_1+N_2)(\|\widehat{P}_m(\lambda)\| + \|\widehat{Q}_m(\lambda)\|) + K\nu^{\varepsilon}_{|m|}.
\end{aligned}
$$

Therefore, for $\lambda \in Y$,

$$
\|\widehat{P}_m(\lambda)\| + \|\widehat{Q}_m(\lambda)\| \le 2K\widehat{K}c(N_1+N_2)(\|\widehat{P}_m(\lambda)\| + \|\widehat{Q}_m(\lambda)|) + K(\mu^{\varepsilon}_{|m|} + \nu^{\varepsilon}_{|m|}).
$$

■

By Lemma 3.5 and Lemma 3.6, (3.5) holds. In order to complete the proof, we only need to show that the stable subspace $\widehat{P}_\lambda(X)$ and the unstable subspace $\widehat{Q}_\lambda(X)$ are Lipschitz continuous in λ.

In fact, from Lemmas 3.1, and 3.2, it follows that U_λ and V_λ are Lipschitz continuous with respect to λ. Note that $\widehat{\mathcal{A}}_\lambda$ is Lipschitz continuous in λ, hence $\widetilde{P}_m(\lambda)$ and $\widetilde{Q}_m(\lambda)$ are Lipschitz continuous in λ. Moreover, since Y is finite-dimensional, $S_0(\lambda)$ and $S_0^{-1}(\lambda)$ are both Lipschitz continuous in λ. Then (3.20) implies that the above claim is valid.

4. Nonlinear perturbations: Grobman-Hartman theorem

In the nonlinear perturbation theory, the linearization of dynamical systems stands as a fundamental step and as a principle tool in the study of local behavior of a given nonlinear flow. The classical Grobman-Hartman theorem, as the well-known linearization theorem, states that, around a hyperbolic fixed point, the map or the flow of a nonlinear dynamical system is topologically conjugate to the corresponding linear map or flow in some open neighborhood of the origin, that is, there exits a homeomorphism such that both maps or flows can be transformed into each other. In this section, with the help of nonuniform (h,k,μ,ν)-dichotomy, we devote to establishing a new version of the Grobman-Hartman theorem for a very general nonuniformly hyperbolic linear operators under nonlinear perturbations.

To facilitate the discussion below, define

$$
\Delta_1 := \left\{ \{u_m\}_{m\in\mathbb{Z}} \in \Delta \;\middle|\; \begin{array}{c} \text{there exist positive constants } l_1 \in \mathbb{R} \text{ and } \omega_1 \in \mathbb{Z} \\ \text{such that any interval of length } l_1 \text{ of } \mathbb{R} \text{ contains at} \\ \text{most } \omega_1 \text{ elements of } \{1/u_m\}_{m\in\mathbb{Z}} \end{array} \right\},
$$

$$
\Delta_2 := \left\{ \{u_m\}_{m\in\mathbb{Z}} \in \Delta \;\middle|\; \begin{array}{c} \text{there exist positive constants } l_2 \in \mathbb{R} \text{ and } \omega_2 \in \mathbb{Z} \\ \text{such that any interval of length } l_2 \text{ of } \mathbb{R} \text{ contains at} \\ \text{most } \omega_2 \text{ elements of } \{u_m\}_{m\in\mathbb{Z}} \end{array} \right\}.
$$

For any constant $\tilde{l} < -1$, $n, m \in \mathbb{Z}$, $l_1 = 1$ and $l_2 = u_n$, one has

$$\sum_{\tau=n}^{\infty} u_\tau^{\tilde{l}} \le \omega_2 u_n^{\tilde{l}} + \omega_2 (2u_n)^{\tilde{l}} + \cdots = \omega_2 u_n^{\tilde{l}} \zeta_{\tilde{l}} \tag{4.1}$$

and

$$\sum_{\tau=-\infty}^{m-1} (u_m/u_\tau)^{\tilde{l}} \le \omega_1 1^{\tilde{l}} + \omega_1 2^{\tilde{l}} + \cdots = \omega_1 \zeta_{\tilde{l}}, \tag{4.2}$$

where $\zeta_{\tilde{l}} := \sum_{\tau=1}^{\infty} \tau^{\tilde{l}}$.

Consider the nonlinear perturbed system of (3.1)

$$x_{m+1} = A_m x_m + f_m(x_m). \tag{4.3}$$

Definition 4.1 ([39, 41])**.** *(3.1) and (4.3) are said to be topologically equivalent if there exist bounded operators $H_m : X \to X, m \in \mathbb{Z}$ with the following properties,*

(i) if $\|x\| \to \infty$, then $\|H_m(x)\| \to \infty$ uniformly with respect to $m \in \mathbb{Z}$;
(ii) for each fixed m, H_m is a homeomorphism of X into X;
(iii) the operators $L_m = H_m^{-1}$ also have property (i);
(iv) if x_m is a solution of (4.3), then $H_m(x_m)$ is a solution of (3.1).

Theorem 4.1. *Assume that*

(c_1) *the sequence of linear operators $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform (h, k, μ, ν)-dichotomy with $|a|, b > 1$ on $\mathbb{Z}$ and $h \in \Delta_1, k \in \Delta_2$;*

(c_2) *there exist positive constants $\hat{\alpha}, \hat{\gamma}$ such that, for any $x, x^1, x^2 \in X$ and $m \in \mathbb{Z}$,*

$$\begin{aligned} &\|f_m(x)\| \le \hat{\alpha} \min\{\mu_{|m+1|}^{-\varepsilon}, \nu_{|m+1|}^{-\varepsilon}\}, \\ &\|f_m(x^1) - f_m(x^2)\| \le \hat{\gamma} \min\{\mu_{|m+1|}^{-\varepsilon}, \nu_{|m+1|}^{-\varepsilon}\} \|x^1 - x^2\|; \end{aligned} \tag{4.4}$$

(c_3) $K\hat{\gamma}(\omega_1 \zeta_a + \omega_2 \zeta_{-b}) < 1$.

Then (4.3) is topologically equivalent to (3.1) and the equivalent operators H_m satisfy

$$\|H_m(x) - x\| \le K\hat{\alpha}(\omega_1 \zeta_a + \omega_2 \zeta_{-b}), \quad m \in \mathbb{Z}, \quad x \in X.$$

In the rest of this section, we always assume that (c_1)-(c_3) are satisfied. Let $X_m(n, x_n)$ be the solution of (4.3) with $X_n = x_n$ and $Y_m(n, y_n)$ be the solution of (3.1) with $Y_n = y_n$. We first prove some auxiliary results.

Lemma 4.1. *For any fixed $(\bar{m}, \xi) \in \mathbb{Z} \times X$,*

(d_1) *the system*

$$z_{m+1} = A_m z_m - f_m(X_m(\bar{m}, \xi)), \; m \in \mathbb{Z} \tag{4.5}$$

has a unique bounded solution $(h_m(\bar{m}, \xi))_{m\in\mathbb{Z}}$ and

$$\|h_m(\bar{m}, \xi)\| \le K\hat{\alpha}(\omega_1 \zeta_a + \omega_2 \zeta_{-b}), \quad m \in \mathbb{Z};$$

(d$_2$) *the system*

$$z_{m+1} = A_m z_m + f_m(Y_m(\bar{m}, \xi) + z_m), \quad m \in \mathbb{Z} \tag{4.6}$$

has a unique bounded solution $(l_m(\bar{m}, \xi))_{m\in\mathbb{Z}}$ *and*

$$\|l_m(\bar{m}, \xi)\| \leq K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b}),\ m \in \mathbb{Z}.$$

Proof. Direct calculations show that

$$\begin{aligned} h_m(\bar{m}, \xi) = &- \sum_{\tau=-\infty}^{m-1} \mathcal{A}(m, \tau+1) P_{\tau+1} f_\tau(X_\tau(\bar{m}, \xi)) \\ &+ \sum_{\tau=m}^{\infty} \mathcal{A}(m, \tau+1) Q_{\tau+1} f_\tau(X_\tau(\bar{m}, \xi)) \end{aligned}$$

is a solution of (4.5). By (4.4), for any $m \in \mathbb{Z}$, one has

$$\begin{aligned} \|h_m(\bar{m}, \xi)\| = &\sum_{\tau=-\infty}^{m-1} \|\mathcal{A}(m, \tau+1) P_{\tau+1}\| \|f_\tau(X_\tau(\bar{m}, \xi))\| \\ &+ \sum_{\tau=m}^{\infty} \|\mathcal{A}(m, \tau+1) Q_{\tau+1}\| \|f_\tau(X_\tau(\bar{m}, \xi))\| \\ \leq &K\hat{\alpha}\left(\sum_{\tau=-\infty}^{m-1} (h_m/h_{\tau+1})^a + k_m^b \sum_{\tau=m}^{\infty} k_{\tau+1}^{-b}\right) \\ \leq &K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b}). \end{aligned}$$

Since the sequence of linear operators $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform (h, k, μ, ν)-dichotomy on $\mathbb{Z}$, $(h_m(\bar{m}, \xi))_{m\in\mathbb{Z}}$ is the unique bounded solution of (4.5).

Set

$$\Omega_3 := \{z : \mathbb{Z} \to X | \|z\| \leq K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b})\},$$

where $\|z\| := \sup_{m\in\mathbb{Z}} \|z_m\|$. It is not difficult to show that $(\Omega_3, \|\cdot\|)$ is a Banach space. Define an operator J on Ω_3 by

$$\begin{aligned} Jz_m = &\sum_{\tau=-\infty}^{m-1} \mathcal{A}(m, \tau+1) P_{\tau+1} f_\tau(Y_\tau(\bar{m}, \xi) + z_\tau) \\ &- \sum_{\tau=m}^{\infty} \mathcal{A}(m, \tau+1) Q_{\tau+1} f_\tau(Y_\tau(\bar{m}, \xi) + z_\tau). \end{aligned}$$

By (c$_2$) and (c$_3$), for any $z, z^1, z^2 \in \Omega_3$ and $m \in \mathbb{Z}$, one has

$$\|Jz_m\| \leq K\hat{\alpha}\left(\sum_{\tau=-\infty}^{m-1} (h_m/h_{\tau+1})^a + k_m^b \sum_{\tau=m}^{\infty} k_{\tau+1}^{-b}\right) \leq K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b})$$

and

$$\begin{aligned}\|Jz_m^1 - Jz_m^2\| &\leq K\hat{\gamma}\left(\sum_{\tau=-\infty}^{m-1}(h_m/h_{\tau+1})^a + k_m^b\sum_{\tau=m}^{\infty}k_{\tau+1}^{-b}\right)\\ &\leq K\hat{\gamma}(\omega_1\zeta_a + \omega_2\zeta_{-b})\|z^1 - z^2\|,\end{aligned}$$

which imply that $J(\Omega_3) \subset \Omega_3$ and J is a contraction. Therefore, J has a unique fixed point $(l_m)_{m\in\mathbb{Z}}$, i.e.,

$$\begin{aligned}l_m(\bar{m},\xi) = &\sum_{\tau=-\infty}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}f_\tau(Y_\tau(\bar{m},\xi)+l_\tau)\\ &-\sum_{\tau=m}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}f_\tau(Y_\tau(\bar{m},\xi)+l_\tau),\end{aligned}$$

which is a bounded solution of (4.6).

Next, we prove that $(l_m(\bar{m},\xi))_{m\in\mathbb{Z}}$ is unique in X. Assume that there is another bounded solution $(l_m^0(\bar{m},\xi))_{m\in\mathbb{Z}}$ of (4.6), which is written as

$$\begin{aligned}l_m^0(\bar{m},\xi) = &\sum_{\tau=-\infty}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}f_\tau(Y_\tau(\bar{m},\xi)+l_\tau^0)\\ &-\sum_{\tau=-\infty}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}f_\tau(Y_\tau(\bar{m},\xi)+l_\tau^0).\end{aligned}$$

Proceeding in a manner similar to the above arguments, we have

$$\|l - l^0\| \leq K\hat{\gamma}(\omega_1\zeta_a + \omega_2\zeta_{-b})\|l - l^0\|.$$

Then, by (c_3), one has $l_m \equiv l_m^0$ for $m \in \mathbb{Z}$. Therefore, $(l_m(\bar{m},\xi))_{m\in\mathbb{Z}}$ is the unique bounded solution of (4.6) with

$$\|l_m(\bar{m},\xi)\| \leq K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b}), \quad m \in \mathbb{Z}.$$

∎

Lemma 4.2. *Let $(x_m)_{m\in\mathbb{Z}}$ be any solution of (4.3). Then $z_m \equiv 0$ is the unique bounded solution of*

$$z_{m+1} = A_m z_m + f_m(x_m + z_m) - f_m(x_m). \tag{4.7}$$

Proof. It is obvious that $z_m \equiv 0$ is a bounded solution of (4.7). Next we show that $z_m \equiv 0$ is unique. Assume that $(z_m^0)_{m\in\mathbb{Z}}$ is any bounded solution of (4.7), then $(z_m^0)_{m\in\mathbb{Z}}$ reads

$$\begin{aligned}z_m^0 = &\sum_{\tau=-\infty}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}[f_\tau(x_\tau+z_\tau) - f_\tau(x_\tau)]\\ &-\sum_{\tau=m}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}[f_\tau(x_\tau+z_\tau) - f_\tau(x_\tau)]\end{aligned}$$

and then $\|z^0 - 0\| \leq K\hat{\gamma}(\omega_1\zeta_a + \omega_2\zeta_{-b})\|z^0 - 0\|$. Therefore, $z_m^0 \equiv 0$. ∎

Define the operators

$$H_m(x) = x + h_m(x), \quad L_m(y) = y + l_m(y), \quad x, y \in X, \ m \in \mathbb{Z}. \tag{4.8}$$

Lemma 4.3. *The following claims hold:*

(e$_1$) *for any fixed* $(\bar{m}, x_{\bar{m}}) \in \mathbb{Z} \times X$, $H_m(X_m(\bar{m}, x_{\bar{m}}))$ *is a solution of* (3.1);

(e$_2$) *for any fixed* $(\bar{m}, y_{\bar{m})} \in \mathbb{Z} \times X$, $L_m(Y_m(\bar{m}, y_{\bar{m}}))$ *is a solution of* (4.3);

(e$_3$) *for any fixed* $m \in \mathbb{Z}$ *and* $y \in X$, $H_m(L_m(y)) = y$ *holds;*

(e$_4$) *for any fixed* $m \in \mathbb{Z}$ *and* $x \in X$, $L_m(H_m(x)) = x$ *holds.*

Proof. From (d$_1$) and (d$_2$) of Lemma 4.1, it follows that

$$h_m(X_m(\bar{m}, x_{\bar{m}})) = h_m(\bar{m}, x_{\bar{m}}), \quad l_m(Y_m(\bar{m}, y_{\bar{m}})) = l_m(\bar{m}, y_{\bar{m}}).$$

Then

$$\begin{aligned} H_m(X_m(\bar{m}, x_{\bar{m}})) &= X_m(\bar{m}, x_{\bar{m}}) + h_m(X_m(\bar{m}, x_{\bar{m}})) \\ &= X_m(\bar{m}, x_{\bar{m}}) + h_m(\bar{m}, x_{\bar{m}}), \\ L_m(Y_m(\bar{m}, y_{\bar{m}})) &= Y_m(\bar{m}, y_{\bar{m}}) + l_m(Y_m(\bar{m}, y_{\bar{m}})) = Y_m(\bar{m}, y_{\bar{m}}) + l_m(\bar{m}, y_{\bar{m}}). \end{aligned}$$

From the fact that $(X_m(\bar{m}, x_{\bar{m}}))_{m\in\mathbb{Z}}$, $(h_m(\bar{m}, x_{\bar{m}}))_{m\in\mathbb{Z}}$, $(Y_m(\bar{m}, y_{\bar{m}}))_{m\in\mathbb{Z}}$, and $l_m(\bar{m}, y_{\bar{m}}))_{m\in\mathbb{Z}}$ are solutions of (4.3), (4.5), (3.1), and (4.6), respectively, it follows that

$$\begin{aligned} H_{m+1}(X_m(\bar{m}, x_{\bar{m}})) &= X_{m+1}(\bar{m}, x_{\bar{m}}) + h_{m+1}(\bar{m}, x_{\bar{m}}) \\ &= A_m X_m(\bar{m}, x_{\bar{m}}) + f_m(X_m(\bar{m}, x_{\bar{m}})) \\ &\quad + A_m h_m(\bar{m}, x_{\bar{m}}) - f_m(X_m(\bar{m}, x_{\bar{m}})) \\ &= A_m H_m(X_m(\bar{m}, x_{\bar{m}})) \end{aligned}$$

and

$$\begin{aligned} L_{m+1}(Y_m(\bar{m}, y_{\bar{m}})) &= Y_{m+1}(\bar{m}, y_{\bar{m}}) + l_{m+1}(\bar{m}, y_{\bar{m}}) \\ &= A_m Y_m(\bar{m}, y_{\bar{m}}) + A_m l_m(\bar{m}, y_{\bar{m}}) \\ &\quad + f_m(Y_m(\bar{m}, y_{\bar{m}}) + l_m(\bar{m}, y_{\bar{m}})) \\ &= A_m L_m(Y_m(\bar{m}, y_{\bar{m}})) + f_m(L_m(Y_m(\bar{m}, y_{\bar{m}}))). \end{aligned}$$

Hence, (e$_1$) and (e$_2$) hold.

Let $(y_m)_{m\in\mathbb{Z}}$ be any solution of (3.1) and $(x_m)_{m\in\mathbb{Z}}$ be any solution of (4.3). It follows from (e$_1$) and (e$_2$) that $(L_m(y_m))_{m\in\mathbb{Z}}$ and $(L_m(H_m(x_m)))_{m\in\mathbb{Z}}$ are solutions of (4.3), $(H_m(L_m(y_m)))_{m\in\mathbb{Z}}$ and $(H_m(x_m))_{m\in\mathbb{Z}}$ are solutions of (3.1). Then

$$\begin{aligned} H_{m+1}(L_m(y_m)) - y_{m+1} &= A_m H_m(L_m(y_m)) - A_m y_m \\ &= A_m(H_m(L_m(y_m)) - y_m) \end{aligned}$$

and

$$
\begin{aligned}
L_{m+1}(H_m(x_m)) - x_{m+1} &= A_m L_m(H_m(x_m)) + f_m(L_m(H_m(x_m))) \\
&\quad - A_m x_m - f_m(x_m) \\
&= A_m(L_m(H_m(x_m)) - x_m) \\
&\quad + f_m(L_m(H_m(x_m)) - x_m + x_m) - f_m(x_m).
\end{aligned}
$$

Moveover,

$$
\begin{aligned}
\|H_m(L_m(y_m)) - y_m\| &\leq \|H_m(L_m(y_m)) - L_m(y_m)\| + \|L_m(y_m) - y_m\| \\
&\leq 2K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b})
\end{aligned}
$$

and

$$
\begin{aligned}
\|L_m(H_m(x_m)) - x_m\| &\leq \|L_m(H_m(x_m)) - H_m(x_m)\| + \|H_m(x_m) - x_m\| \\
&\leq 2K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b}).
\end{aligned}
$$

Therefore, $(H_m(L_m(y_m))-y_m)_{m\in\mathbb{Z}}$ is a bounded solution of (3.1) and $H_m(L_m(y_m))-y_m \equiv 0$. For any fixed $m \in \mathbb{Z}$ and $y \in X$, there exists a solution of (3.1) with the initial value $y_m = y$. Then $H_m(L_m(y)) = y$ for any $m \in \mathbb{Z}$. On the other hand, by Lemma 4.2, we conclude that $L_m(H_m(x_m)) - x_m \equiv 0$ for any $m \in \mathbb{Z}$. For any fixed $m \in \mathbb{Z}$ and $x \in X$, there exists a solution of (4.3) with the initial value $x_m = x$. Then $L_m(H_m(x)) = x$ holds for any $m \in \mathbb{Z}$.
∎

In order to establish Theorem 4.1, we only need to verify that $(H_m)_{m\in\mathbb{Z}}$ are topologically equivalent operators. In fact,

- Condition (i): it follows from (4.8) and (d_1) of Lemma 4.1 that

$$
\|H_m(x) - x\| = \|h_m(x)\| \leq K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b}), \ m \in \mathbb{Z}, \ x \in X.
$$

  Then $\|H_m(x)\| \to \infty$ uniformly with respect to $m \in \mathbb{Z}$ as $\|x\| \to \infty$;

- Condition (ii): by (e_3) and (e_4) of Lemma 4.3, for each fixed $m \in \mathbb{Z}$, $H_m = L_m^{-1}$ is homeomorphism;

- Condition (iii): by (4.8), for any $m \in \mathbb{Z}$,

$$
\|L_m(y) - y\| = \|l_m(y)\| \leq K\hat{\alpha}(\omega_1\zeta_a + \omega_2\zeta_{-b}), \ m \in \mathbb{Z}, \ y \in X.
$$

  This implies that $\|L_m(y)\| \to \infty$ uniformly with respect to $m \in \mathbb{Z}$ as $\|y\| \to \infty$;

- Condition (iv): it follows from Lemma 4.3 that the condition (iv) holds.

5. Nonlinear perturbations: parameter dependence of stable Lipschitz invariant manifolds

It has been widely recognized that, both in mathematics and in application, the classical theory of invariant manifolds provides the geometric structures for describing and understanding the qualitative behavior of nonlinear dynamical systems. In this section, we establish the existence of parameter dependence of stable Lipschitz invariant manifolds for sufficiently small nonlinear perturbations of (3.1) with the nonuniform (h, k, μ, ν)-dichotomy. Since here we only consider the case of stable invariant manifold, then we only need to carry out the discussion on $\mathbb{Z}^+$.

Consider the nonlinear perturbed system with the parameters of (3.1)

$$x_{m+1} = A_m x_m + f_m(x_m, \lambda), \tag{5.1}$$

where $f_m : X \times Y \to X$ and $f_m(0, \lambda) = 0$ for any $m \in \mathbb{Z}^+$ and $\lambda \in Y$. In order to establish the existence of stable invariant manifolds and for convenience of the discussion, we rewrite the nonuniform (h, k, μ, ν)-dichotomy in the following equivalent form

$$\begin{aligned} \|\mathcal{A}(m, n)P_n\| &\le K\,(h_m/h_n)^a\,\mu_n^\varepsilon, \\ \|\mathcal{A}(m, n)^{-1}Q_m\| &\le K\,(k_m/k_n)^{-b}\,\nu_m^\varepsilon \end{aligned} \tag{5.2}$$

for $m \ge n$ and define the stable and unstable spaces by $E_m = P_m(X), F_m = Q_m(X), m \in \mathbb{Z}^+$, respectively.

Let

$$\beta_m = k_m^{b/(\varepsilon q)} h_m^{-a(q+1)/(\varepsilon q)} \mu_m^{1+1/q} C_m^{1/(\varepsilon q)}, \tag{5.3}$$

where

$$C_m = \sum_{\tau=m}^{\infty} h_\tau^{aq} (h_\tau/h_{\tau+1})^a \max\{\mu_{\tau+1}^\varepsilon, \nu_{\tau+1}^\varepsilon\},$$

and $B_n(\varrho) \subset E_n$ be the open ball centered at zero with radius ϱ for a given $n \in \mathbb{Z}^+$.

Denote by $\mathcal{X}$ the space of sequences of operators $\Phi_n\colon Z_\beta = Z_\beta(1) \to X$ satisfying

$$\Phi_n(0) = 0, \quad \Phi_n(B_n(\beta_n^{-\varepsilon})) \subset F_n,$$

and

$$\|\Phi_n(\xi_1) - \Phi_n(\xi_2)\| \le \|\xi_1 - \xi_2\| \tag{5.4}$$

for any $n \in \mathbb{Z}^+$ and $\xi_1, \xi_2 \in B_n(\beta_n^{-\varepsilon})$, where

$$Z_\beta(\eta) = \big\{(n, \xi) : n \in \mathbb{Z}^+,\ \xi \in B_n(\beta_n^{-\varepsilon}/\eta)\big\}$$

and η is a positive constant. It is not difficult to show that $\mathcal{X}$ is a Banach space with the norm

$$|\Phi|' = \sup\left\{\frac{\|\Phi_n(\xi)\|}{\|\xi\|} : n \in \mathbb{Z}^+ \text{ and } \xi \in B_n(\beta_n^{-\varepsilon}) \setminus \{0\}\right\}.$$

On the other hand, let $\mathcal{X}^*$ be the space of sequences of operators $\Phi_n\colon \mathbb{Z}^+ \times X \to X$ such that $\Phi|_{Z_\beta} \in \mathcal{X}$ and $\Phi_n(\xi) = \Phi_n\big(\beta_n^{-\varepsilon}\xi/\|\xi\|\big), (n, \xi) \notin Z_\beta$. It is clear that there is a

one-to-one correspondence between $\mathcal{X}$ and $\mathcal{X}^*$ and $\mathcal{X}^*$ is a Banach space with the norm $\mathcal{X}^* \ni \Phi \mapsto |\Phi|Z_\beta|'$. For $n \in \mathbb{Z}^+$ and $\xi_1, \xi_2 \in E_n$, one has

$$\|\Phi_n(\xi_1) - \Phi_n(\xi_2)\| \le 2\|\xi_1 - \xi_2\|. \tag{5.5}$$

For $\lambda \in Y$ and $(n, u_n, v_n) \in \mathbb{Z}^+ \times E_n \times F_n$, consider the graph

$$\mathcal{W}_\lambda = \big\{(n, \xi, \Phi_n(\xi)) : (n, \xi) \in Z_\beta, \Phi \in \mathcal{X}\big\} \tag{5.6}$$

and

$$\Psi_\kappa^\lambda(n, u_n, v_n) = (m, u_m, v_m), \quad \kappa = m - n \ge 0, \tag{5.7}$$

where

$$u_m = \mathcal{A}(m,n)u_n + \sum_{\tau=n}^{m-1} \mathcal{A}(m, \tau+1)P_{\tau+1}f_\tau(u_\tau, v_\tau, \lambda), \tag{5.8}$$

$$v_m = \mathcal{A}(m,n)v_n + \sum_{\tau=n}^{m-1} \mathcal{A}(m, \tau+1)Q_{\tau+1}f_\tau(u_\tau, v_\tau, \lambda). \tag{5.9}$$

We now establish the existence of a stable Lipschitz invariant manifold for (5.1).

Theorem 5.1. *Assume that*

(g_1) *there exist positive constants $\hat{c}$ and q such that*

$$\|f_m(x^1, \lambda) - f_m(x^2, \lambda)\| \le \hat{c}\|x^1 - x^2\|(\|x^1\|^q + \|x^2\|^q) \tag{5.10}$$

and

$$\|f_m(x, \lambda_1) - f_m(x, \lambda_2)\| \le \hat{c}|\lambda_1 - \lambda_2| \cdot \|x\|^{q+1} \tag{5.11}$$

for any $m \in \mathbb{Z}^+$, $x, x^1, x^2 \in X$ and $\lambda, \lambda_1, \lambda_2 \in Y$;

(g_2) *the sequence of linear operators $(A_m)_{m\in\mathbb{Z}^+}$ admits a nonuniform (h,k,μ,ν)-dichotomy;*

(g_3) $\lim_{m\to\infty} k_m^{-b} h_m^a \nu_m^\varepsilon = 0$ *and $h_m^a \beta_m^\varepsilon$ is a decreasing sequence.*

If $\hat{c}$ in (5.10) and (5.11) is sufficiently small, then, for each $\lambda \in Y$, for any (n,ξ), (n, ξ_1), $(n, \xi_2) \in Z_{\beta\cdot\mu}(2K)$ and $\kappa = m - n \ge 0$, there exist a unique sequence of operators $\Phi_n = \Phi_n^\lambda \in \mathcal{X}$ and a constant $d > 0$ such that

$$\Psi_\kappa^\lambda(n, \xi, \Phi_n(\xi)) \in \mathcal{W}_\lambda \tag{5.12}$$

and

$$\|\Psi_\kappa^\lambda(n, \xi_1, \Phi_n(\xi_1)) - \Psi_\kappa^\lambda(n, \xi_2, \Phi_n(\xi_2))\| \le d(h_m/h_n)^a \mu_n^\varepsilon \|\xi_1 - \xi_2\|. \tag{5.13}$$

Moreover, there exists a constant $d^* > 0$ such that

$$\|\Psi_\kappa^{\lambda_1}(n, \xi, \Phi_n^{\lambda_1}(\xi)) - \Psi_\kappa^{\lambda_2}(n, \xi, \Phi_n^{\lambda_2}(\xi))\| \le d(h_m/h_n)^a \mu_n^\varepsilon |\lambda_1 - \lambda_2| \cdot \|\xi\|. \tag{5.14}$$

for any $\lambda_1, \lambda_2 \in Y$.

Proof. We first prove that, for each $(n,\xi,\Phi,\lambda)\in Z_\beta\times\mathcal{X}^*\times Y$, there exists a unique sequence of operators $u=u_\xi^{\Phi,\lambda}:\mathbb{Z}^+\to X$ with $u_n=\xi$ such that (5.8) holds for any $m\ge n$ and

$$\|u_m\|\le 2K(h_m/h_n)^a\mu_n^\varepsilon\|\xi\|. \tag{5.15}$$

Let

$$\Omega_3:=\{u\colon[n,\infty)\to X|\|u\|_*\le\beta_n^{-\varepsilon},u_m\in E_m,u_n=\xi,m\ge n\},$$

where

$$\|u\|_*=\frac{1}{2K}\sup\left\{\frac{\|u_m\|}{(h_m/h_n)^a\mu_n^\varepsilon}:m\ge n\right\}. \tag{5.16}$$

Then Ω_3 is a Banach space with the norm $\|\cdot\|_*$. Given $(n,\xi)\in Z_\beta$ and $\Phi\in\mathcal{X}^*$, for each $\lambda\in Y$, define an operator L^λ on Ω_3 by

$$L^\lambda u_m=\mathcal{A}(m,n)\xi+\sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)P_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda).$$

Obviously, $L^\lambda u_n=\xi$ and $L^\lambda u_m\in E_m$ for $m\ge n$. By (5.10) and (5.2), one has

$$\begin{aligned}B_\tau^{\lambda,1}:&=\|f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda)\|\\&\le\hat{c}\left(\|u_\tau\|+\|\Phi_\tau(u_\tau)\|\right)\left(\|u_\tau\|+\|\Phi_\tau(u_\tau)\|\right)^q\\&\le 3^{q+1}\hat{c}\|u_\tau\|^{q+1}\\&\le 6^{q+1}\hat{c}K^{q+1}\left(\frac{h_\tau}{h_n}\right)^{a(q+1)}\mu_n^{\varepsilon(q+1)}(\|u\|_*)^{q+1},\ \tau\ge n\end{aligned}$$

and

$$\begin{aligned}\|L^\lambda u_m\|&\le\|\mathcal{A}(m,n)\|\|\xi\|+\sum_{\tau=n}^{m-1}\|\mathcal{A}(m,\tau+1)P_{\tau+1}\|B_\tau^{\lambda,1}\\&\le K\left(\frac{h_m}{h_n}\right)^a\mu_n^\varepsilon\|\xi\|+6^{q+1}\hat{c}K^{q+2}\left(\frac{h_m}{h_n}\right)^a h_n^{-aq}\mu_n^{\varepsilon(q+1)}(\|u\|_*)^{q+1}C_n.\end{aligned}$$

Then

$$\begin{aligned}\|L^\lambda u\|_*&\le\frac12\left(\|\xi\|+6^{q+1}\hat{c}K^{q+1}h_n^{-aq}\mu_n^{\varepsilon q}(\|u\|_*)^{q+1}C_n\right)\\&\le\frac12\left(1+6^{q+1}\hat{c}K^{q+1}h_n^{-aq}\mu_n^{\varepsilon q}\beta_n^{-\varepsilon q}C_n\right)\beta_n^{-\varepsilon}\\&\le\frac12(1+6^{q+1}\hat{c}K^{q+1})\beta_n^{-\varepsilon}.\end{aligned}$$

Hence, $L^\lambda(\Omega_3)\subset\Omega_3$ since $\hat{c}$ is sufficiently small and one can take a $\hat{c}$ such that $6^{q+1}\hat{c}K^{q+1}<1$. Moreover, for any $u^1,u^2\in\Omega_3$, it follows that

$$\begin{aligned}B_\tau^{\lambda,2}:&=\|f_\tau(u_\tau^1,\Phi_\tau(u_\tau^1),\lambda)-f_\tau(u_\tau^2,\Phi_\tau(u_\tau^2),\lambda)\|\\&\le 3^{q+1}\hat{c}\|u_\tau^1-u_\tau^2\|(\|\mu_\tau^1\|^q+\|\mu_\tau^2\|^q)\\&\le 2^{q+2}3^{q+1}\hat{c}K^{q+1}\left(\frac{h_\tau}{h_n}\right)^{a(q+1)}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon q}\|u^1-u^2\|_*\end{aligned}$$

and

$$\begin{aligned}\|L^\lambda u_m^1 - L^\lambda u_m^2\| &\le \sum_{\tau=n}^{m-1} \|\mathcal{A}(m,\tau+1)P_{\tau+1}\| B_\tau^{\lambda,2} \\ &\le 2\cdot 6^{q+1}\hat{c}K^{q+2}\|u^1-u^2\|_* \left(\frac{h_m}{h_n}\right)^a \mu_n^\varepsilon.\end{aligned}$$

Then $\|L^\lambda u^1 - L^\lambda u^2\|_* \le 6^{q+1}\hat{c}K^{q+1}\|u^1-u^2\|_*$. Since $\hat{c}$ is sufficiently small, take $\hat{c}$ such that $6^{q+1}\hat{c}K^{q+1} < 1$, then L^λ is a contraction in Ω_3 and there exists a unique sequence of operators $u = u^\lambda \in \Omega_3$ such that $L^\lambda u = u$. On the other hand, since $K/(1-(1/2)6^{q+1}\hat{c}K^{q+1}) < 2K$, it is not difficult to show that, for any $m \ge n$,

$$\|u\|_* \le \frac{1}{2}\|\xi\| + \frac{1}{2}6^{q+1}\hat{c}K^{q+1}\|u\|_*, \quad \|u_m\| \le 2K(h_m/h_n)^a\mu_n^\varepsilon\|\xi\|.$$

Next we study the properties of the unique sequence of operators $u = u_\xi^{\Phi,\lambda}$.

For each $\lambda \in Y$ and $\Phi \in \mathcal{X}^*$, write $u^i = u_{\xi_i}^{\Phi,\lambda}$ for $i = 1,2$ and $(n,\xi_1),(n,\xi_2) \in Z_\beta$. By (5.5) and (5.10), one has

$$\begin{aligned}B_\tau^{\lambda,3} :&= \|f_\tau(u_\tau^1,\Phi_\tau(u_\tau^1),\lambda) - f_\tau(u_\tau^2,\Phi_\tau(u_\tau^2),\lambda)\| \\ &\le 3^{q+1}\hat{c}\|u_\tau^1-u_\tau^2\|(\|u_\tau^1\|^q + \|u_\tau^2\|^q).\end{aligned}$$

Then

$$\begin{aligned}\|u_m^1-u_m^2\| &\le \|\mathcal{A}(m,n)(\xi_1-\xi_2)\| + \sum_{\tau=n}^{m-1}\|\mathcal{A}(m,\tau+1)P_{\tau+1}\|B_\tau^{\lambda,3} \\ &\le K\left(\frac{h_m}{h_n}\right)^a \mu_n^\varepsilon(\|\xi_1-\xi_2\| \\ &\qquad + 2\cdot 6^{q+1}\hat{c}K^{q+2}\|u^1-u^2\|_*\left(\frac{h_m}{h_n}\right)^a \mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon q}C_n\end{aligned}$$

and

$$\|u^1-u^2\|_* \le \frac{1}{2}\|\xi_1-\xi_2\| + 6^{q+1}\hat{c}K^{q+1}\|u^1-u^2\|_*.$$

Therefore,

$$\|u_m^1-u_m^2\| \le K_1(h_m/h_n)^a\mu_n^\varepsilon\|\xi_1-\xi_2\| \tag{5.17}$$

with $K_1 = K/(1-6^{q+1}\hat{c}K^{q+1})$ if $\hat{c}$ is sufficiently small.

For each $\lambda \in Y$ and each $(n,\xi) \in Z_\beta$, write $u^i = u_\xi^{\Phi^i,\lambda}$ for $i = 1,2$ and $\Phi^1,\Phi^2 \in \mathcal{X}^*$. With the help of (5.2), (5.5), (5.10), and (5.15), one has

$$\begin{aligned}B_\tau^{\lambda,4} :&= \|f_\tau(u_\tau^1,\Phi_\tau^1(u_\tau^1),\lambda) - f_\tau(u_\tau^2,\Phi_\tau^2(u_\tau^2),\lambda)\| \\ &\le 3^q\hat{c}\left[3(\|u_\tau^1-u_\tau^2\|)(\|u_\tau^1\|^q+\|u_\tau^2\|^q)\cdot(\|u_\tau^1\|\cdot|\Phi^1-\Phi^2|')(\|u_\tau^1\|^q+\|u_\tau^2\|^q)\right] \\ &\le [2\cdot 6^{q+1}\hat{c}K^{q+1}\|u^1-u^2\|_* + 4\cdot 6^q\hat{c}K^{q+1}\|\xi\|\cdot|\Phi^1-\Phi^2|'] \\ &\quad \times (h_\tau/h_n)^{a(q+1)}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon q},\ \tau \ge n\end{aligned}$$

and

$$\begin{aligned}\|u_m^1-u_m^2\| &\le \sum_{\tau=n}^{m-1}\|\mathcal{A}(m,\tau+1)P_{\tau+1}\|B_\tau^{\lambda,4}\\ &\le [2\cdot 6^{q+1}\hat{c}K^{q+1}\|u^1-u^2\|_*+4\cdot 6^q\hat{c}K^{q+1}\|\xi\|\cdot|\Phi^1-\Phi^2|']\\ &\quad\times K\left(\frac{h_m}{h_n}\right)^a h_n^{-aq}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon q}C_n.\end{aligned}$$

Then

$$\|u^1-u^2\|_*\le[6^{q+1}\hat{c}K^{q+1}\|u^1-u^2\|^*+2\cdot 6^q\hat{c}K^{q+1}\|\xi\|\cdot|\Phi^1-\Phi^2|']\mu_n^{-\varepsilon}$$

and

$$\|u_m^1-u_m^2\|\le K_2(\mu_m/\mu_n)^a\|\xi\|\cdot|\Phi^1-\Phi^2|' \tag{5.18}$$

with $K_2=4\cdot 6^q\hat{c}K^{q+2}/(1-6^{q+1}\hat{c}K^{q+1})$.

In order to establish the existence and uniqueness of the sequence of operators $\Phi_n=\Phi_n^\lambda\in\mathcal{X}$ satisfying (5.9) for each given $\lambda\in Y$, we will prove that, if $\hat{c}$ is sufficiently small and $\Phi_n\in\mathcal{X}^*$, then one has the following claims:

(h$_1$) for $(n,\xi)\in Z_\beta$ and $m\ge n$, if

$$\Phi_m(u_m)=\mathcal{A}(m,n)\Phi_n(\xi)+\sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda), \tag{5.19}$$

then

$$\Phi_n(\xi)=-\sum_{\tau=n}^{\infty}\mathcal{A}(\tau+1,n)^{-1}Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda); \tag{5.20}$$

(h$_2$) if (5.20) holds for $n\in\mathbb{Z}^+$ and $\xi\in B_n(\beta_n^{-\varepsilon})$, then (5.19) holds for $(n,\xi)\in Z_{\beta\cdot\mu}(2K)$.

It follows from (5.2), (5.10), (5.5) and (5.15) that

$$\begin{aligned}B_\tau^{\lambda,5}:&=\|\mathcal{A}(\tau+1,n)^{-1}Q_{\tau+1}\|\cdot\|f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda)\|\\ &\le 3^{q+1}\hat{c}K\left(\frac{k_{\tau+1}}{k_n}\right)^{-b}\nu_{\tau+1}^\varepsilon\|u_\tau\|^{q+1}\\ &\le 6^{q+1}\hat{c}K^{q+2}\left(\frac{k_{\tau+1}}{k_n}\right)^{-b}\nu_{\tau+1}^\varepsilon\left(\frac{h_\tau}{h_n}\right)^{a(q+1)}\mu_n^{\varepsilon(q+1)}\|\xi\|^{q+1}\\ &\le 6^{q+1}\hat{c}K^{q+2}\left(\frac{k_{\tau+1}}{k_n}\right)^{-b}\nu_{\tau+1}^\varepsilon\left(\frac{h_\tau}{h_n}\right)^{a(q+1)}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon(q+1)}\end{aligned}$$

and

$$\begin{aligned}\sum_{\tau=n}^{\infty}B_\tau^{\lambda,5}&\le 6^{q+1}\hat{c}K^{q+2}k_n^bh_n^{-a(q+1)}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon(q+1)}\sum_{\tau=n}^{\infty}k_{\tau+1}^{-b}h_\tau^{a(q+1)}\nu_\tau^\varepsilon\\ &\le 6^{q+1}\hat{c}K^{q+2}k_n^bh_n^{-a(q+1)}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon q}C_n<\infty.\end{aligned}$$

Then the right-hand side of (5.20) is well-defined. If (5.19) holds for $(n,\xi)\in Z_\beta$ and $m\geq n$, then we rewrite (5.19) as

$$\Phi_n(\xi)=\mathcal{A}(m,n)^{-1}\Phi_m(u_m)-\sum_{\tau=n}^{m-1}\mathcal{A}(\tau+1,n)^{-1}Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda). \tag{5.21}$$

By (5.2), (5.5), and (5.15), it follows that

$$\begin{aligned}\|\mathcal{A}(m,n)^{-1}\Phi_m(u_m)\| &\leq 4K^2\left(\frac{k_m}{k_n}\right)^{-b}\nu_m^\varepsilon\left(\frac{h_m}{h_n}\right)^{a}\mu_n^\varepsilon\beta_n^{-\varepsilon}\\ &\leq 4K^2k_m^{-b}h_m^a\nu_m^\varepsilon k_n^b h_n^{-a}\mu_n^\varepsilon\beta_n^{-\varepsilon}.\end{aligned}$$

Therefore, letting $t\to\infty$ in (5.21) yields (5.20). On the other hand, assume that (5.20) holds for any $(n,\xi)\in Z_\beta$, then, for $(n,\xi)\in Z_{\beta\cdot\mu}(2K)$,

$$\|u_m\|\leq 2K\left(\frac{h_m}{h_n}\right)^{a}\mu_n^\varepsilon\|\xi\|\leq\beta_m^{-\varepsilon}\frac{h_m^a\beta_m^\varepsilon}{h_n^a\beta_n^\varepsilon}\leq\beta_m^{-\varepsilon}.$$

Hence, $(m,u_m)\in Z_\beta$ for any $m\geq n$. By (5.20), one gets

$$\begin{aligned}\mathcal{A}(m,n)\Phi_n(\xi)&=-\sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda)\\ &\quad-\sum_{\tau=m}^{\infty}\mathcal{A}(m,\tau+1)Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda)\\ &=-\sum_{\tau=n}^{m-1}\mathcal{A}(m,\tau+1)Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda)+\Phi_m(\mu_m),\end{aligned}$$

where we have used (5.20) in the last equality with (n,ξ) replaced by (m,u_m).

We now state the existence and uniqueness of the sequence of operators $\Phi_n=\Phi_n^\lambda\in\mathcal{X}$ such that (5.12) and (5.13) hold for each $\lambda\in Y$.

Given $\lambda\in Y$, for $\Phi_n\in\mathcal{X}^*$ and $(n,\xi)\in Z_\beta$, define an operator J^λ by

$$(J^\lambda\Phi_n)(\xi)=-\sum_{\tau=n}^{\infty}\mathcal{A}(\tau+1,n)^{-1}Q_{\tau+1}f_\tau(u_\tau,\Phi_\tau(u_\tau),\lambda),$$

where u is the unique sequence of operators in (5.15) for (n,ξ,Φ,λ). Moreover, we have $J^\lambda\Phi_n(0)=0$ and

$$\begin{aligned}B_\tau^{\lambda,6}:&=\|f_\tau(u_\tau^1,\Phi_\tau(u_\tau^1),\lambda)-f_\tau(u_\tau^2,\Phi_\tau(u_\tau^2),\lambda)\|\\ &\leq 3^{q+1}\hat{c}\|u_\tau^1-u_\tau^2\|(\|u_\tau^1\|^q+\|u_\tau^2\|^q)\\ &\leq 6^{q+1}\hat{c}K^qK_1\left(\frac{h_\tau}{h_n}\right)^{a(q+1)}\mu_n^{\varepsilon(q+1)}\beta_n^{-\varepsilon q}\|\xi_1-\xi_2\|\end{aligned}$$

for any $\xi_1, \xi_2 \in B_n(\beta_n^{-\varepsilon})$ and $u^i = u_{\xi_i}^{\Phi,\lambda}$ for $i = 1, 2$. Then

$$\begin{aligned}\|J^\lambda \Phi_n(\xi_1) - J^\lambda \Phi_n(\xi_2)\| &\le \sum_{\tau=n}^{\infty} \|\mathcal{A}(\tau+1,n)^{-1} Q_{\tau+1}\| B_\tau^{\lambda,6} \\ &\le 6^{q+1} \hat{c} K^{q+1} K_1 k_n^b h_n^{-a(q+1)} \mu_n^{\varepsilon(q+1)} \beta_n^{-\varepsilon q} C_n \|\xi_1 - \xi_2\| \\ &\le 6^{q+1} \hat{c} K^{q+1} K_1 \|\xi_1 - \xi_2\|\end{aligned}$$

and

$$\|J^\lambda \Phi_n(\xi_1) - J^\lambda \Phi_n(\xi_2)\| \le \|\xi_1 - \xi_2\|$$

since $\hat{c}$ is sufficiently small. It is not difficult to extend $J^\lambda \Phi$ to $\mathbb{Z}^+ \times X$ by $J^\lambda \Phi_n(\xi) = J^\lambda \Phi_n\big(\beta_n^{-\varepsilon} \xi / \|\xi\|\big)$ for any $(n, \xi) \notin Z_\beta$, and hence, $J^\lambda(\mathcal{X}^*) \subset \mathcal{X}^*$. For any $\Phi^1, \Phi^2 \in \mathcal{X}^*$, writing $u^i = u_\xi^{\Phi^i,\lambda}$ for $i = 1, 2$, by (5.5), (5.15), and (5.18), for each $(n, \xi) \in Z_\beta$, we have

$$\begin{aligned}B_\tau^{\lambda,7} :&= \|f_\tau(u_\tau^1, \Phi_\tau^1(u_\tau^1), \lambda) - f_\tau(u_\tau^2, \Phi_\tau^2(u_\tau^2), \lambda)\| \\ &\le 3^q \hat{c}\big(3\|u_\tau^1 - u_\tau^2\| + \|u_\tau^1\| \cdot |\Phi^1 - \Phi^2|'\big)(\|u_\tau^1\|^q + \|u_\tau^2\|^q) \\ &\le 2 \cdot 6^q \hat{c} K^q (2K + 3K_2) \|\xi\| \cdot |\Phi^1 - \Phi^2|' \cdot \left(\frac{h_\tau}{h_n}\right)^{a(q+1)} \mu_n^{\varepsilon(q+1)} \beta_n^{-\varepsilon q}\end{aligned}$$

and

$$\begin{aligned}\|J^\lambda \Phi_n^1(\xi) - J^\lambda \Phi_n^2(\xi)\| &\le \sum_{\tau=n}^{\infty} \|\mathcal{A}(\tau+1,n)^{-1} Q_{\tau+1}\| B_\tau^{\lambda,7} \\ &\le 2 \cdot 6^q \hat{c} K^q (2K + 3K_2) \|\xi\| \cdot |\Phi_1 - \Phi_2|'.\end{aligned}$$

Therefore, the operator J^λ is a contraction for each $\lambda \in Y$ and there exists a unique sequence of operators $\Phi = \Phi^\lambda \in \mathcal{X}^*$ such that (5.20) holds for every $(n, \xi) \in Z_\beta$. From (h_2) and the one-to-one correspondence between $\mathcal{X}$ and $\mathcal{X}^*$, it follows that there exists a unique sequence of operators $\Phi = \Phi^\lambda \in \mathcal{X}$ such that (5.19) holds for $\lambda \in Y$ and $n \in \mathbb{Z}^+$, $\xi \in B_n\left((\beta_n \cdot \mu_n)^{-\varepsilon}/(2K)\right)$. For each $(n, \xi) \in Z_{\beta\cdot\mu}(2K)$, by (5.15), we have

$$\|u_m\| \le 2K (h_m/h_n)^a \mu_n^\varepsilon \frac{1}{2K} (\beta_n \cdot \mu_n)^{-\varepsilon} \le (h_m/h_n)^a \beta_n^{-\varepsilon} \le \beta_n^{-\varepsilon},$$

which implies that $(m, u_m) \in Z_\beta$ for any $m \ge n$. Therefore, (5.12) holds. For any $(n, \xi_1), (n, \xi_2) \in Z_{\beta\cdot\mu}(2K)$, $\lambda \in Y$, and $\kappa = m - n \ge 0$, we have

$$\begin{aligned}&\|\Psi_\kappa^\lambda(n, \xi_1, \Phi_n(\xi_1)) - \Psi_\kappa^\lambda(n, \xi_2, \Phi_n(\xi_2))\| \\ &= \|(m, u_m^{\xi_1,\lambda}, \Phi_m(u_m^{\xi_1,\lambda}) - (m, u_m^{\xi_2,\lambda}, \Phi_m(u_m^{\xi_2,\lambda}))\| \\ &\le 3\|u_m^{\xi_1,\lambda} - u_m^{\xi_2,\lambda}\| \le 3K_1 (h_m/h_n)^a \mu_n^\varepsilon \|\xi_1 - \xi_2\|.\end{aligned}$$

To complete the proof, one only needs to establish the inequality (5.14). For $(n, \xi) \in Z_{\beta\cdot\mu}(2K)$ and $\lambda_1, \lambda_2 \in Y$, set $u^{\lambda_1} = u_\xi^{\Phi^{\lambda_1},\lambda_1}, u^{\lambda_2} = u_\xi^{\Phi^{\lambda_2},\lambda_2}$, by (5.2), (5.5), (5.10), (5.11), (5.15), (5.17) and (5.18), one has

$$
\begin{aligned}
B_\tau^{\lambda,8} :&= \|f_\tau(u_\tau^{\lambda_1}, \Phi_\tau^{\lambda_1}(u_\tau^{\lambda_1}), \lambda_1) - f_\tau(u_\tau^{\lambda_2}, \Phi_\tau^{\lambda_2}(u_\tau^{\lambda_2}), \lambda_2)\| \\
&\le \|f_\tau(u_\tau^{\lambda_1}, \Phi_\tau^{\lambda_1}(u_\tau^{\lambda_1}), \lambda_1) - f_\tau(u_\tau^{\lambda_1}, \Phi_\tau^{\lambda_1}(u_\tau^{\lambda_1}), \lambda_2)\| \\
&\quad + \|f_\tau(u_\tau^{\lambda_1}, \Phi_\tau^{\lambda_1}(u_\tau^{\lambda_1}), \lambda_2) - f_\tau(u_\tau^{\lambda_2}, \Phi_\tau^{\lambda_2}(u_\tau^{\lambda_2}), \lambda_2)\| \\
&\le 6^{q+1} K^{q+1} \hat{c} (h_\tau/h_n)^{a(q+1)} \mu_n^{\varepsilon(q+1)} \\
&\quad \times [|\lambda_1 - \lambda_2| \cdot \|\xi\|^{q+1} + 2\|\xi\|^q \|u^{\lambda_1} - u^{\lambda_2}\|_* + \frac{2}{3}|\Phi^{\lambda_1} - \Phi^{\lambda_2}|' \cdot \|\xi\|^{q+1}]
\end{aligned}
$$

and

$$
\begin{aligned}
\|\Phi_n^{\lambda_1}(\xi) - \Phi_n^{\lambda_2}(\xi)\| &\le \sum_{\tau=n}^{\infty} \|\mathcal{A}(\tau+1, n)^{-1} Q_{\tau+1}\| B_\tau^{\lambda,8} \\
&\le h'|\lambda_1 - \lambda_2| \cdot \|\xi\| + 2h' \|u^{\lambda_1} - u^{\lambda_2}\|_* \\
&\quad + (2/3) h' |\Phi^{\lambda_1} - \Phi^{\lambda_2}|' \cdot \|\xi\|,
\end{aligned}
$$

where $h' = 2 \cdot 3^{q+1} K^2 \hat{c}$, which implies that, if $\hat{c}$ is sufficiently small, then

$$
|\Phi^{\lambda_1} - \Phi^{\lambda_2}|' \le H|\lambda_1 - \lambda_2| + 2H \|u^{\lambda_1} - u^{\lambda_2}\|_* / \|\xi\|
$$

and

$$
\|\Phi_n^{\lambda_1}(\xi) - \Phi_n^{\lambda_2}(\xi)\| \le H|\lambda_1 - \lambda_2| \cdot \|\xi\| + 2H \|u^{\lambda_1} - u^{\lambda_2}\|_*,
$$

where $H = h'/(1 - (2/3)h')$. Whence,

$$
\begin{aligned}
\|u_m^{\lambda_1} - u_m^{\lambda_2}\| &\le \sum_{\tau=n}^{m-1} \|\mathcal{A}(m, \tau+1) P_{\tau+1}\| B_\tau^{\lambda,8} \\
&\le h' \left((1 + (2/3)H) |\lambda_1 - \lambda_2| \cdot \|\xi\| \right. \\
&\quad \left. + (2 + (4/3)H) \|u^{\lambda_1} - u^{\lambda_2}\|_* \right) (h_m/h_n)^a \mu_n^\varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\|u^{\lambda_1} - u^{\lambda_2}\|_* &\le [\overline{H}/(2K)] |\lambda_1 - \lambda_2| \cdot \|\xi\| \cdot \|u_m^{\lambda_1} - u_m^{\lambda_2}\| \\
&\le \overline{H} (h_m/h_n)^a \mu_n^\varepsilon |\lambda_1 - \lambda_2| \cdot \|\xi\|
\end{aligned}
$$

where $\overline{H} = h'(1 + (2/3)H)/(1 - h'(1 + (2/3)H)/K)$. Therefore, for $(n, \xi) \in Z_{\beta \cdot \mu}(2K)$, $\lambda_1, \lambda_2 \in Y$ and $\kappa = m - n \ge 0$, we have

$$
\begin{aligned}
&\|\Psi_\kappa^{\lambda_1}(n, \xi, \Phi_n^{\lambda_1}(\xi)) - \Psi_\kappa^{\lambda_2}(n, \xi, \Phi_n^{\lambda_2}(\xi))\| \\
&= \|(m, u_m^{\lambda_1}, \Phi_m^{\lambda_1}(u_m^{\lambda_1})) - (m, u_m^{\lambda_2}, \Phi_m^{\lambda_2}(u_m^{\lambda_2}))\| \\
&\le \|u_m^{\lambda_1} - u_m^{\lambda_2}\| + \|\Phi_m^{\lambda_1}(u_m^{\lambda_1}) - \Phi_m^{\lambda_2}(u_m^{\lambda_2})\| \\
&\le \|u_m^{\lambda_1} - u_m^{\lambda_2}\| + \|\Phi_m^{\lambda_1}(u_m^{\lambda_1}) - \Phi_m^{\lambda_1}(u_m^{\lambda_2})\| + \|\Phi_m^{\lambda_1}(u_m^{\lambda_2}) - \Phi_m^{\lambda_2}(u_m^{\lambda_2})\| \\
&\le 3\|u_m^{\lambda_1} - u_m^{\lambda_2}\| + |\Phi^{\lambda_1} - \Phi^{\lambda_2}|' \|u_m^{\lambda_2}\| \\
&\le [3\overline{H} + 2KH(1 + \overline{H}/K)] (h_m/h_n)^a \mu_n^\varepsilon \|\lambda_1 - \lambda_2| \cdot \|\xi\|,
\end{aligned}
$$

which implies that (5.14) holds. The proof is complete. ∎

Acknowledgement

The authors would like to thank Prof. Ken Palmer for his suggestive discussion and nice comments and to thank Prof. Luis Barreira for leading them into the field of the nonuniform hyperbolicity and the nonuniform dichotomies.

References

[1] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z. 32 (1930) 703-728.

[2] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, Berlin-New York, 1978.

[3] A.M. Fink, Almost Periodic Differential Equations, in: Lecture Notes in Mathematics, vol. 377, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[4] P.E. Kloeden, M. Rasmussen, Nonautonomous Dynamical Systems, The American Mathematical Society Publishers, 2011.

[5] C.V.M. Mee, Exponentially Dichotomous Operators and Applications, Birkhäuser Basel Publishers, 2008.

[6] L. Barreira, C. Valls, Stability of Nonautonomous Differential Equations, Lecture Notes in Math., vol. 1926, Springer-Verlag, Berlin-New York, 2008.

[7] R. Naulin, M. Pinto, Roughness of (h, k)-dichotomies, J. Differ. Equations, 118 (1995) 20-35.

[8] M. Megan, B. Sasu, A. Sasu, On nonuniform exponential dichotomy of evolution operators in Banach spaces, Integr. Equ. Oper. Theory, 44 (2002) 71-78.

[9] A. Minda, M. Megan, On (h, k)-stability of evolution operators in Banach spaces, Appl. Math. Let. 24 (2011) 44-48.

[10] C. Preda, P. Preda, A. Craciunescu, A version of a theorem of R. Datko for nonuniform exponential contractions, J. Math. Anal. Appl. 385 (2012) 572-581.

[11] N. Lupa, M. Megan, Exponential dichotomies of evolution operators in Banach spaces, Monatsh. Math. 2013, DOI 10.1007/s00605-013-0517-y.

[12] L.P. Jiang, Generalized exponential dichotomy and global linearization, J. Math. Anal. Appl. 315 (2006) 474-490.

[13] A. Bento, C. Silva, Stable manifolds for nonuniform polynomial dichotomies, J. Funct. Anal. 257 (2009) 122-148.

[14] A. Bento, C. Silva, Nonuniform (μ, ν)-dichotomies and local dynamics of difference equations, Nonlinear Analysis TMA 75 (2012) 78-90.

[15] A. Bento, C. Silva, Generalized nonuniform dichotomies and local stable manifolds, J. Dynam. Differ. Equations, 25 (2013) 1139-1158.

[16] A.L. Sasu, M.G. Babuţia, B. Sasu, Admissibility and nonuniform exponential dichotomy on the half-line, Bull. Sci. Math. 137 (2013) 466-484.

[17] J.F. Chu, F.F. Liao, S. Siegmund, Y.H. Xia, W.N. Zhang, Nonuniform dichotomy spectrum and reducibility for nonautonomous equations, Bull. Sci. Math. in press, 2014.

[18] L. Barreira, Ya. Pesin, Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents, Cambridge Univ. Press, 2007.

[19] J. Graczyk, S. Smirnov, Non-uniform hyperbolicity in complex dynamics, Invent. Math. 175 (2009) 335-415.

[20] Y. Pesin, V. Climenhaga. Open problems in the theory of non-uniform hyperbolicity, Discrete Conti. Dyn. S. 27 (2010) 55-76.

[21] M. Shub, A. Wilkinson, Pathological foliations and removable zero exponents, Invent. Math. 139 (2000) 495-508.

[22] L. Barreira, C. Valls, Polynomial growth rates, Nonlinear Analysis TMA 71 (2009) 5208-5219.

[23] L. Barreira, M. Fan, C. Valls, J.M. Zhang, Robustness of nonuniform polynomial dichotomies for difference equations, Topol. Method. Nonl. An. 37 (2011) 357-376.

[24] L. Barreira, C. Valls, Robustness of general dichotomies, J. Funct. Anal. 257 (2009) 464-484.

[25] X.Y. Chang, J.M. Zhang, J.H. Qin, Robustness of nonuniform (μ, ν)-dichotomies in Banach spaces, J. Math. Anal. Appl. 387 (2012) 582-594.

[26] L. Barreira, J. Chu, C. Valls, Lyapunov functions for general nonuniform dichotomies, Milan J. Math. 81 (2013) 153-169.

[27] J.F. Chu, Robustness of nonuniform behavior for discrete dynamics, Bull. Sci. Math. 137 (2013) 1031-1047.

[28] N. Ju, S. Wiggins, On roughness of exponential dichotomy, J. Math. Anal. Appl. 262 (2001) 39-49.

[29] R. Naulin, M. Pinto, Admissible perturbations of exponential dichotomy roughness, Nonlinear Analysis TMA 31 (1998) 559-571.

[30] R. Naulin, M. Pinto, Stability of discrete dichotomies for linear difference systems, J. Differ. Equ. Appl. 3 (1997) 101-123.

[31] L. Barreira, D. Dragičević, C. Valls, Lyapunov functions for strong exponential dichotomies, J. Math. Anal. Appl. 399 (2013) 116-132.

[32] S.N. Chow, H. Leiva, Existence and roughness of the exponential dichotomy for skew-product semiflows in Banach spaces, J. Differ. Equations, 120 (1995) 429-477.

[33] O. Méndez, L. Popescu, On admissible perturbations for exponential dichotomy, J. Math. Anal. Appl. 337 (2008) 425-430.

[34] V. Pliss, G. Sell, Robustness of exponential dichotomies in infinite-dimensional dynamical systems, J. Dynam. Differ. Equations, 11 (1999) 471-513.

[35] L. Popescu, Exponential dichotomy roughness and structural stability for evolution families without bounded growth and decay, Nonlinear Analysis TMA 71 (2009) 935-947.

[36] L. Popescu, Exponential dichotomy roughness on Banach spaces, J. Math. Anal. Appl. 314 (2006) 436-454.

[37] L. Barreira, C. Valls, Robust nonuniform dichotomies and parameter dependence, J. Math. Anal. Appl. 373 (2011) 690-708.

[38] L. Barreira, C. Valls, A Grobman-Hartman theorem for general nonuniform exponential dichotomies, J. Funct. Anal. 257 (2009) 1976-1993.

[39] J. Kurzweil, Topological equivalence and structural stability for linear difference equations, J. Differ. Equations, 89 (1991) 89-94.

[40] K.J. Palmer, A generalization of Hartman's linearization theorem, J. Math. Anal. Appl. 41 (1973) 753-758.

[41] K.J. Palmer, The structurally stable linear systems on the half-line are those with exponential dichotomies, J. Differ. Equations 33 (1979) 16-25.

[42] G. Papaschinopoulos, J. Schinas, Structural stability via the density of a class of linear discrete systems, J. Math. Anal. Appl. 127 (1987) 530-539.

[43] L. Popescu, A topological classification of linear differential equations on Banach spaces, J. Differ. Equations 203 (2004) 28-37.

[44] Y.H. Xia. R.T. Wang, K.I. Kou, D. O'Regan, On the linearization theorem for nonautonomous differential equations, Bull. Sci. Math. in press, 2015.

[45] L. Barreira, M. Fan, C. Valls, J.M. Zhang, Stable manifold for delay equations and parameter dependence, Nonlinear Analysis TMA 75 (2012) 5824-5835.

[46] L. Barreira, M. Fan, C. Valls, J.M. Zhang, Parameter dependence of stable manifolds for delay equations with polynomial dichotomies, J. Dynam. Differ. Equations, 24 (2012)101-118.

[47] L. Barreira, C. Valls, Parameter dependence of stable manifolds for difference equations, Nonlinearity, 23 (2010) 341-367.

[48] S.N. Chow, X.B. Lin, K. Lu, Smooth invariant foliations in infinite dimensional spaces. J. Differ. Equations, 94 (1991) 266-291.

[49] D. Filipović, J. Teichmann, Existence of invariant manifolds for stochastic equations in infinite dimension, J. Funct. Anal. 197 (2003) 398-432.

[50] R.J. Sacker, G.R. Sell, The spectrum of an invariant submanifold, J. Differ. Equations, 38 (1980) 135-160.

[51] C.V. Coffman, J.J. Schäffer, Dichotomies for linear difference equations, Math. Ann. 172 (1967) 139-166.

[52] G. Papaschinopoulos, J. Schinas, Criteria for an exponential dichotomy of difference equations, Czech. Math. J. 35 (1985) 295-299.

[53] A. Bento, C. Silva, Nonuniform dichotomic behavior:Lipschitz invarant manifolds for difference eqautions, arXiv math/1209.6589 [math.DS] 2012.